12. Vorlesung 11.12.20

Beweis: $\sum a_n$, $a_n \neq 0$ für alle n

$$\lim \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$\Longrightarrow \quad \frac{|a_{n+1}|}{|a_n|} \le q < 1, \quad n \ge N$$

Dann gilt:

$$|a_{n+1}| \le q \cdot |a_n|, \quad n \ge N$$

Induktion:

$$|a_n| \le q^{n-N} |a_N|$$

$$= c \cdot q^n, \quad c = q^{-N} |a_N|.$$

Also:

$$\sum c q^n \quad \text{konv. Majorante.}$$

Fall $\lim \ldots > 1$,

dann

$$\left| \frac{a_{n+1}}{a_n} \right| \ge 1, \quad n \ge N$$

Also $a_n \not\to 0$. Divergent. ∎

Bsp:

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$z \in \mathbb{C}$: $z = 0$ ✓ Sei $z \neq 0$:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right|$$

$$= \left| \frac{z}{n+1} \right| \longrightarrow 0, \quad n \to \infty$$

Also absolut konvergent für alle $z \in \mathbb{C}$.

Beweis: $k \geq 0$:

$$\underbrace{\sum_{\dots} a_{2^{k+1}} \leq \overbrace{\sum_{2^k < n \leq 2^{k+1}} a_n}^{2^k \text{ Summanden}} \leq \sum_{\dots} a_{2^k}}$$

$$2^k \cdot a_{2^{k+1}} \leq \sum_{2^k < n \leq 2^{k+1}} a_n \leq 2^k \cdot a_{2^k}$$

Summe für $k = 1, \dots, n$:

$$\boxed{\sum_{k=1}^{n} 2^k a_{2^{k+1}} \leq \underbrace{\sum_{k=2}^{2^{n+1}} a_k}_{\text{div}} \leq \underbrace{\sum_{k=1}^{n} 2^k a_{2^k}}_{\text{div}}}$$

div $\longrightarrow$ div .

$\blacksquare$

$\alpha > 0$:

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

div. für $\alpha \leq 1$

konv. für $\alpha > 1$.

WK: $\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{n^\alpha}} = \frac{1}{(\sqrt[n]{n})^\alpha} \longrightarrow 1$

QK: $\frac{a_{n+1}}{a_n} = \frac{n^\alpha}{(n+1)^\alpha} = \left(\frac{n}{n+1}\right)^\alpha \longrightarrow 1.$

Verdichtete Reihe: $\sum \frac{1}{n^\alpha}$ $\quad n \sim 2^n$

$$\sum 2^n \cdot \frac{1}{(2^n)^\alpha} = \sum 2^{n-\alpha n}$$

$$= \sum \underbrace{(2^{1-\alpha})^n}_{\text{geom. Reihe}}$$

Konvergent für

$2^{1-\alpha} < 1 \iff \alpha > 1$

Div für

$2^{1-\alpha} \geq 1 \iff \alpha \leq 1.$

Beweis: Es gilt für alle n große

$$\frac{a_{n+1}}{a_n} \le 1-\frac{\alpha}{n}, \qquad n \ge n_0.$$

$$\Longleftrightarrow \qquad a_{n+1} \le \left(1-\frac{\alpha}{n}\right) a_n$$

Induktion:

$$a_{n+1} \le a_{n_0} \prod_{k=n_0}^{n} \left(1-\frac{\alpha}{k}\right).$$

Also:

$$\underbrace{a_{n+1}} \le c \cdot \underbrace{\prod_{k=1}^{n} \left(1-\frac{\alpha}{k}\right)} \le \frac{c}{n^\alpha}.$$

mit $c>0$ hinr. groß.

Bew:

$$\prod_{k=1}^{n} \left(1-\frac{\alpha}{k}\right) \le \frac{1}{n^\alpha}, \qquad n \ge 1.$$

Induktion: $n=1$: $1-\frac{\alpha}{1} \le 1$ ✓

Induktionsschritt: $n-1 \Rightarrow n$

$$\frac{1}{(n-1)^\alpha}\left(1-\frac{\alpha}{n}\right) \le \frac{1}{n^\alpha}$$

$$\Longleftrightarrow \qquad 1-\frac{\alpha}{n} \le \underbrace{\left(\frac{n-1}{n}\right)^\alpha = \left(1-\frac{1}{n}\right)^\alpha}_{\text{Bernoulli!}}$$ ✓

Bsp:

$$\sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots \cdot (2n+1)}{2\cdot 4\cdot \ldots \cdot (2n+2)}$$

$$\frac{a_{n+1}}{a_n} = \frac{2n+1}{2n+4} \xrightarrow[n\to\infty]{} 1$$

$$= 1 - \frac{3}{2n+4}$$

$$\leq 1 - \underbrace{\frac{4}{3}}_{\alpha}\cdot\frac{1}{n}, \qquad n \geq 16.$$

Prova: Parti dispari:

$$S_{2n+1} - S_{2n-1} = a_{2n} - a_{2n+1} \geq 0$$

$$S_{2n+2} - S_{2n} = a_{2n+2} - a_{2n+1} \leq 0$$

$$S_{2n+2} - S_{2n+1} = a_{2n+2} \geq 0 \longrightarrow 0$$

Allora:

$$S_1 \leq S_3 \leq S_5 \leq \dots \qquad S_6 \leq S_4 \leq S_2 \leq S_0$$

$$\left.\begin{aligned} S_{2n-1} &\longrightarrow S_- \\ S_{2n} &\longrightarrow S_+ \end{aligned}\right\} =$$

$\square$

Es: Armonica Alt. Resto

$$\sum_{n\geq 1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} \mp \dots$$

Converge, poiché $\frac{1}{n} \searrow 0$.

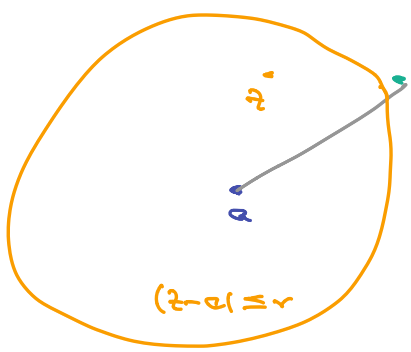
$= \log 2$.

$\square$

$$\sum_{n\geq 0} a_n (z-a)^n$$

a_n, a, z in $\mathbb{C}$

a_n, a, z in $\mathbb{R}$ nella pratica



z_0 : $f(z_0)$ conv.

$r = |z_0 - a| > 0$

Έστω: $\varphi(z_0)$ συγκλίνει:

$$a_n |z_0 - a|^n \longrightarrow 0$$

Άρα:

$$\sup_{n \geq 0} |a_n (z_0 - a)|^n \leq M < \infty$$

Άρα:

$$|a_n| \leq \frac{M}{|z_0 - a|^n}, \quad n \geq 0.$$

Έστω τώρα z με $|z - a| \leq \underbrace{r < |z_0 - a|}$.

Τότε:

$$|a_n (z-a)^n| \leq |a_n| \cdot r^n$$

$$\leq M \cdot \frac{r^n}{|z_0 - a|^n}$$

Θέτω

$$\rho = \frac{r}{|z_0 - a|} \underbrace{< 1}$$

για κάθε n:

$$|a_n (z-a)^n| \leq M \cdot \rho^n$$

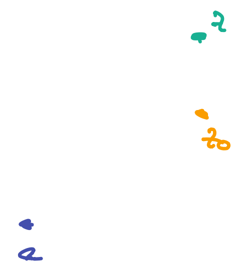
Άρα γεωμ. κριτήριο:

$$\varphi(z) = \sum a_n (z-a) \quad \text{και} \quad \sum M \rho^n$$

& Βασ. κριτήριο.

Παρατήρηση:

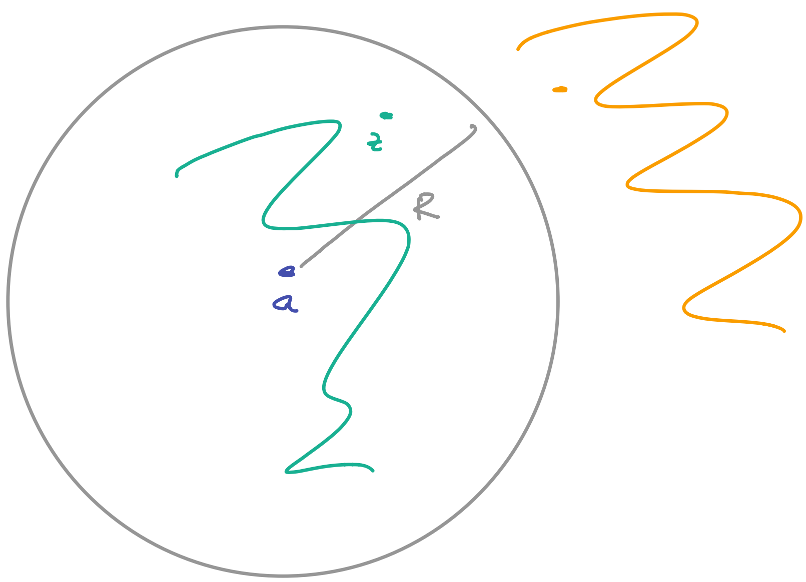
ίδιο ρ για όλα τα z με $|z - a| \leq r$.



$f(z_0)$ diverges

$|z-a| > |z_0-a|$

then Div.



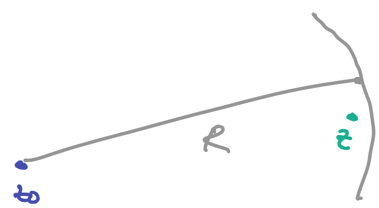
Beweis:

$$K = \{ z \in \mathbb{C} : \phi(z) \text{ ist kompakt} \}$$

Dann $\quad z_0 \in K, \qquad K \neq \emptyset.$

$$R = \sup \{ |z - z_0| : z \in K \} \subseteq \mathbb{R}$$

Also: $\quad R \in [0, \infty].$



$$|z - z_0| < R$$

Dann $\exists\ z_1 \in K$:

$$|z_1 - z_0| > |z - z_0|$$

Bedeutung R:

(i) $R = 0$: konvergent nur für $z = a$.

"triviale Fall".

(ii) $R = \infty$: konvergent für alle $z \in \mathbb{C}$.

(iii) $0 < R < \infty$:

Bem.: Es gibt Formel für R:

$$R = \frac{1}{\underbrace{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}}$$

wobei $\frac{1}{0} = \infty$ und $\frac{1}{\infty} = 0$

$R = \infty$ $\qquad$ $R = 0$

$$\phi(z) = \sum_{n\geq 0} z^n \quad , \quad z \in \mathbb{C}$$

$$\sqrt[n]{|z^n|} = |z|$$

$$R = 1$$

$$\underbrace{\phi(z) = \sum_{n\geq 0} z^n}_{|z|<1} = \underbrace{\frac{1}{1-z}}_{\text{défini si } z\in\mathbb{C}\setminus\{1\}} .$$

$a \neq 1$:

$$\frac{1}{1-z} = \frac{1}{(1-a)-(z-a)}$$

$$= \frac{1}{1-a} \cdot \frac{1}{1-\frac{z-a}{1-a}}$$

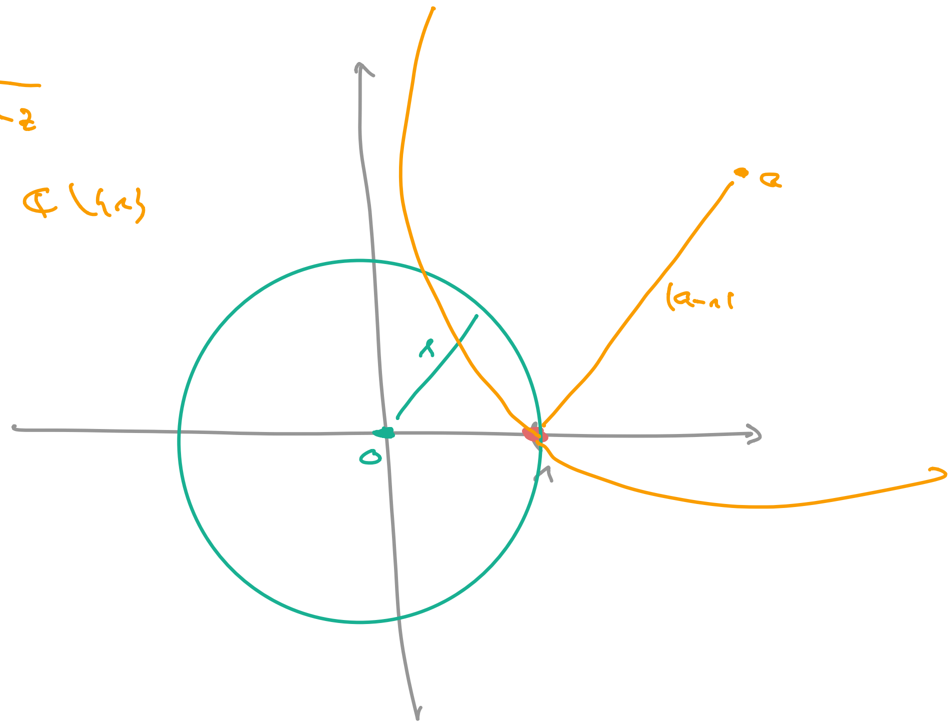
$$= \frac{1}{1-a} \cdot \sum_{n\geq 0} \left(\frac{z-a}{1-a}\right)^n$$

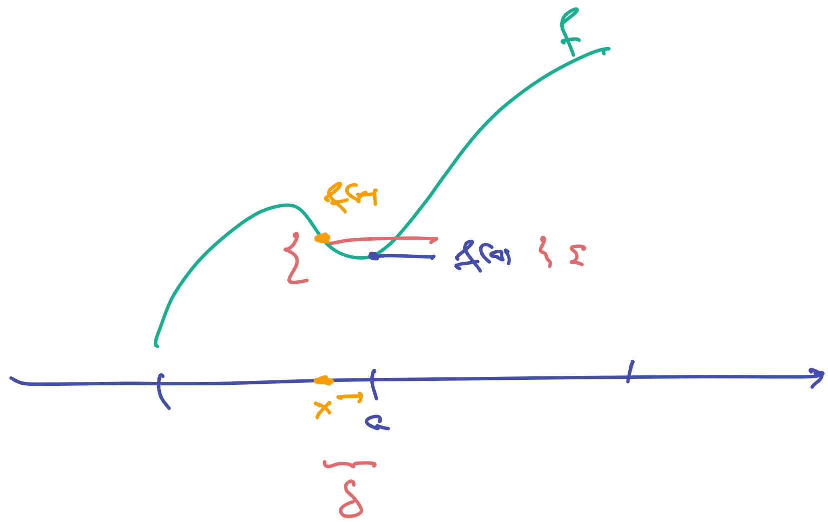
$$= \sum_{n\geq 0} \frac{1}{(1-a)^{n+1}} (z-a)^n$$

$$R = |a-1|$$

$$\frac{1}{1-z}$$

auf $\mathbb{C}\setminus\{1\}$





$$" \quad x \to a \quad \Rightarrow \quad f(x) \to f(a) \quad "$$

$$|x-a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \varepsilon$$

$$f : D \to \mathbb{R}, \qquad D \subset \mathbb{R}$$

Kiive:

$$f : \mathbb{R} \supset D \to \mathbb{R}$$

[illegible] D [illegible] [illegible].

$$f : \mathbb{R} \supset D \longrightarrow \mathbb{R}$$

$$a \in D$$

Zu jedem $\varepsilon > 0$ ein $\delta > 0$:

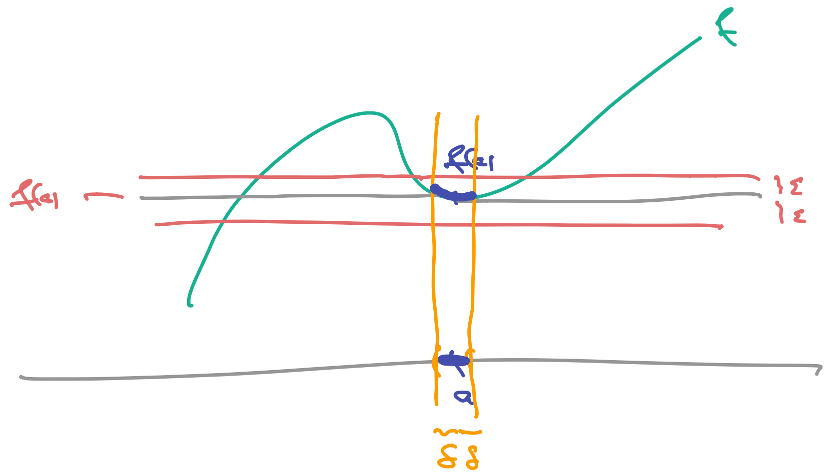
$$x \in D \wedge |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon$$

Zu jedem $\varepsilon > 0$ ex. $\delta > 0$:

$$x \in U_\delta(a) \cap D \Longrightarrow f(x) \in U_\varepsilon(f(a))$$

Kurz

$$f\left(U_\delta(a) \cap D\right) \subset U_\varepsilon(f(a))$$



Bsp.: $f(t) = t^2 \quad , \quad t \in \mathbb{R}$

$a \in \mathbb{R}$:

$$|f(t) - f(a)| = |t^2 - a^2|$$

$$= |t-a| \cdot |t+a| \overset{!}{<} \varepsilon$$

$$\Leftarrow \quad |t-a| < \frac{\varepsilon}{|t+a|}$$

Für $|t-a| \le 1$ gilt : $|t+a| \le 2|a|+1$:

$$\Leftarrow \quad |t-a| < \frac{\varepsilon}{2|a|+1} = \delta(\varepsilon, a) .$$

$\square$