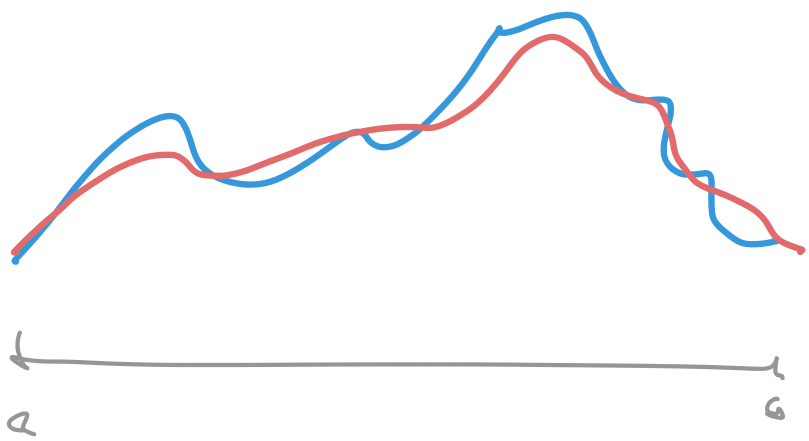
8. Vorlesung 11.5.2021



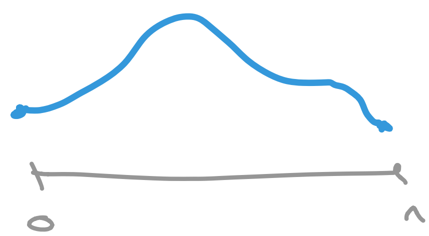
Remark: Generality:

Let f defined on $[a,b]$

$$u : [0,1] \longrightarrow [a,b]$$

$$u(t) = (1-t)a + tb$$

$$g = f \circ u : [0,1] \longrightarrow \mathbb{R}$$



$$\lambda : [0,1] \longrightarrow \mathbb{R}$$

$$\lambda(t) = (1-t)\, g(0) + t\, g(1)$$

$$h = g - \lambda : [0,1] \longrightarrow \mathbb{R}$$

$$h(0) = 0, \quad h(1) = 0.$$

$$P_n \Longrightarrow h \quad \text{on } [0,1]$$

Then:

$$(P_n + \lambda) \circ u^{-1} \Longrightarrow f$$

since it

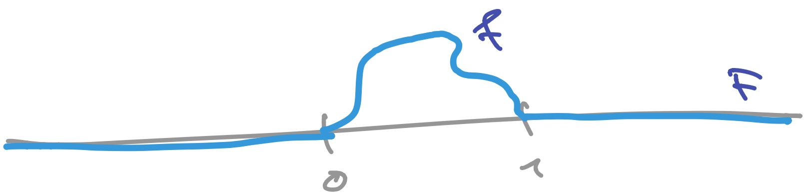
Poly in t.

Gegeben $f : [0,1] \to \mathbb{R}$,

$$f(0) = f(1) = 0.$$

Definiere

$$F := \begin{cases} f & \text{auf } [0,1] \\ 0 & \text{sonst.} \end{cases}$$



Definiere:

$$\varphi_n(t) = \frac{1}{\lambda_n}(1-t^2)^n, \qquad |t| \le 1$$

$$\lambda_n := \int_{-1}^{1} (1-t^2)^n \, dt$$

Also:

$$\int_{-1}^{1} \varphi_n(t)\, dt = 1, \qquad \varphi_n \ge 0.$$

Wichtige Idee: (φ_n) ist Dirac-Folge: s.u.

$$\Phi_n := \begin{cases} \varphi_n & \text{auf } [-1,1] \\ 0 & \text{sonst.} \end{cases}$$

$$p_n = f * \Phi_n .$$

Dirac folge

glm. stetig auf $\mathbb{R}$

Also:

$$p_n \rightrightarrows f .$$

Also auch

$$p_n\big|_{[0,1]} \rightrightarrows f\big|_{[0,1]} = f .$$

Bem: p_n sind Polynome!

Denn $x \in [0,1]$:

$$f * \Phi_n(x) = \int_{-\infty}^{\infty} f(t)\, \Phi_n(x-t)\, dt$$

$$= \int_0^1 f(t)\, \Phi_n(x-t)\, dt$$

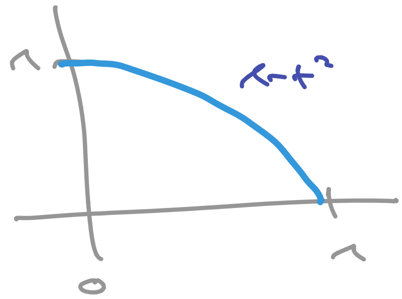
$$= \frac{1}{\lambda_n} \cdot \int_0^1 f(t)\, \underbrace{\left(1-(x-t)^2\right)^n}_{\sum_{k=0}^{2n} a_k(t) \cdot x^k}\, dt$$

$$= \underbrace{\frac{1}{\lambda_n} \sum_{k=0}^{2n} \left(\int_0^1 f(t)\, a_k(t)\, dt \right) \cdot x^k}_{\text{Polynom in } x.}$$

$\blacksquare$

Beweis des Lemmas:

$1-t^2$ fällt auf $[0,1]$ monoton:



$$\int_\delta^1 \underbrace{(1-t^2)^n}_{t=\delta}\,dt \le (1-\delta^2)^n .$$

$$\begin{aligned}\lambda_n &\ge \int_0^1 (1-t^2)^n\,dt \\ &= \int_0^1 (1-t)^n \underbrace{(1+t)^n}_{\ge 1}\,dt \\ &\ge \int_0^1 (1-t)^n\,dt = \frac{1}{n+1} .\end{aligned}$$

Also:

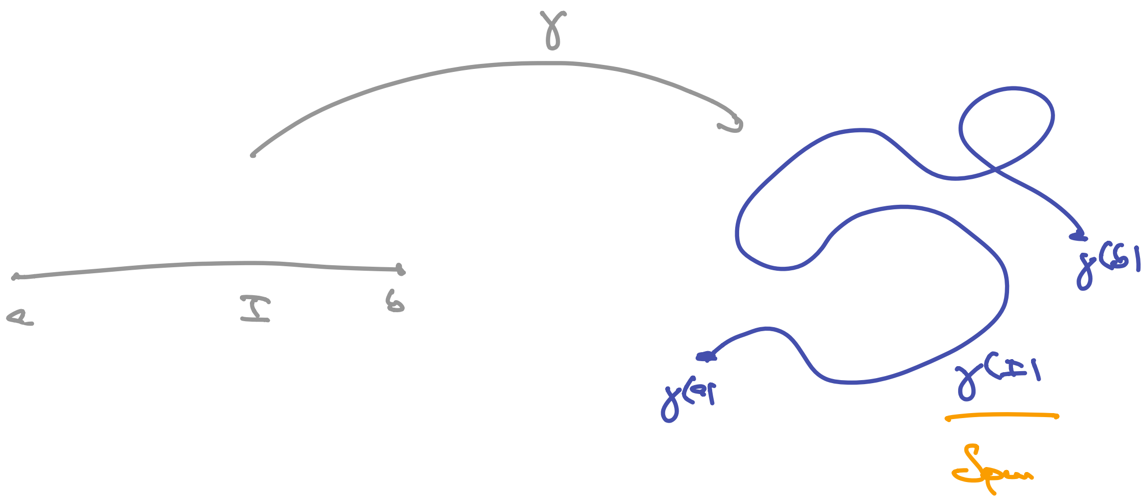
$$\frac{1}{\lambda_n} \le n+1 .$$

Also,

$$\int_\delta^1 \varphi_n(t)\,dt \le (n+1)\underbrace{(1-\delta^2)}_{=q<1}{}^n \xrightarrow[n\to\infty]{} 0 .$$

$\blacksquare$

Kurven und Wege



$\gamma : [a,b] \longrightarrow E$,

$\gamma(a)$ Anfangs-
$\gamma(b)$ Endpunkt von γ.

$\gamma(a) = \gamma(b)$: geschlossen

Beispiele:

1. $p \in E$: Konstante Kurve

$\gamma: I \longrightarrow E$, $\gamma(t) = p$.

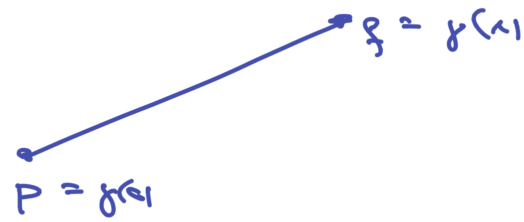
geschlossen, aber nicht doppelpunktfrei, falls $\ell(I) > 0$.

2. $p, q \in E$

$\gamma: [0,1] \longrightarrow E$

$\gamma(t) = (1-t)p + tq$

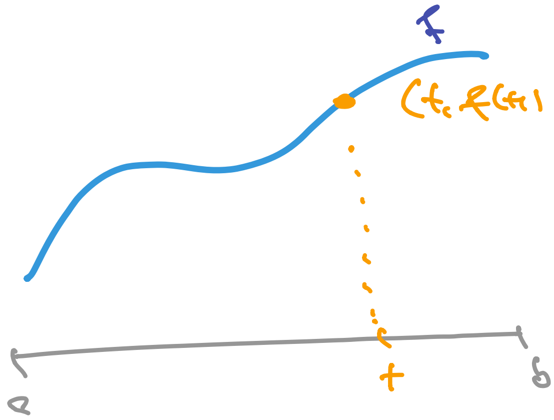
Strecke von p nach q:



$p \neq q$: doppelpunktfrei, nicht geschlossen.

$p = q$: konstant, geschlossen, nicht doppelpunktfrei.

3.



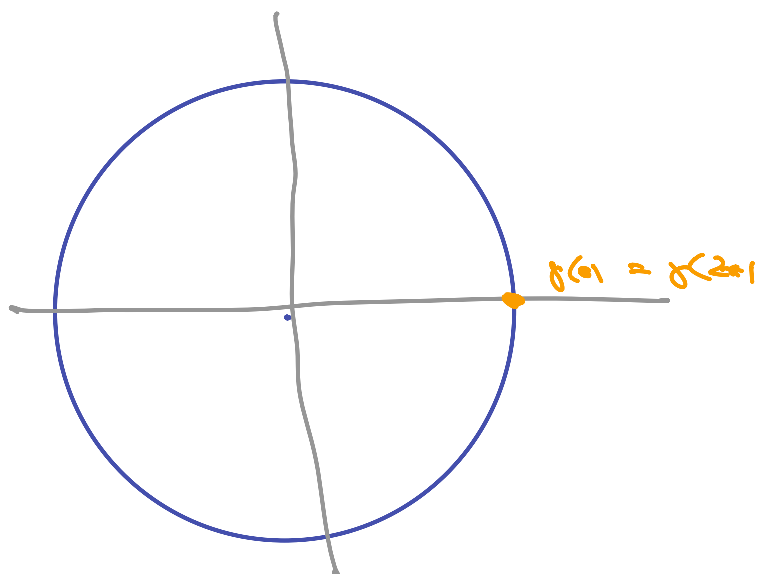
$$\gamma : I = [a, b] \longrightarrow \mathbb{R}^2$$

$$\gamma(t) = (t, f(t))$$

immer doppelpunktfrei,

nicht geschlossen.

4.



$$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$$

$$\gamma(t) = (\cos t, \sin t)$$

geschlossen, doppelpunktfrei.

5. Der n-mal durchlaufene Einheitskreis:

$$\gamma_n : [0, 2\pi] \longrightarrow \mathbb{R}^2$$

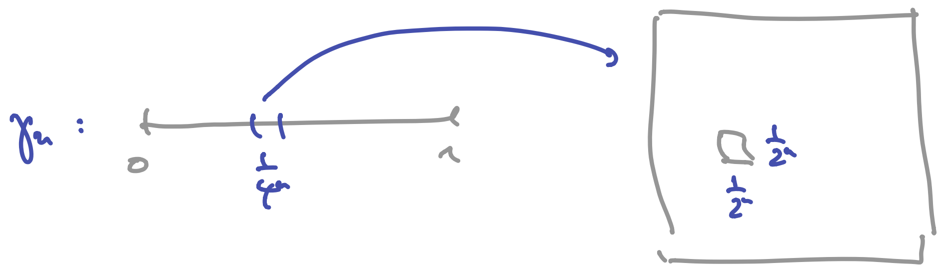
$$\gamma_n(t) = (\cos nt, \ \sin nt), \quad n \in \mathbb{Z}.$$

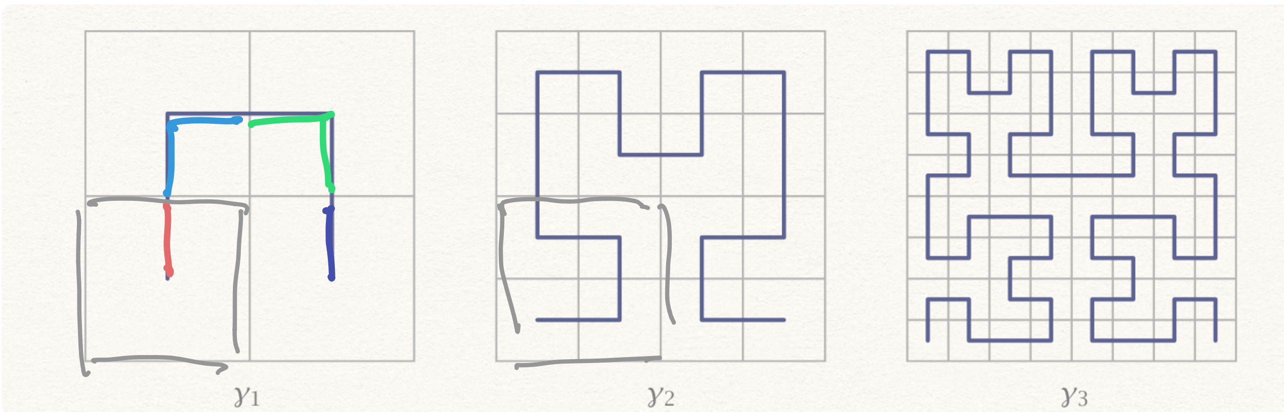
$n = 1$: $\gamma_1 = \gamma$

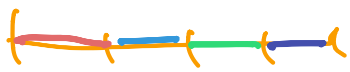
glatt, aber für $|n| \neq 1$

nicht doppelpunktfrei

$$\gamma_0(t) \equiv (1, 0)$$







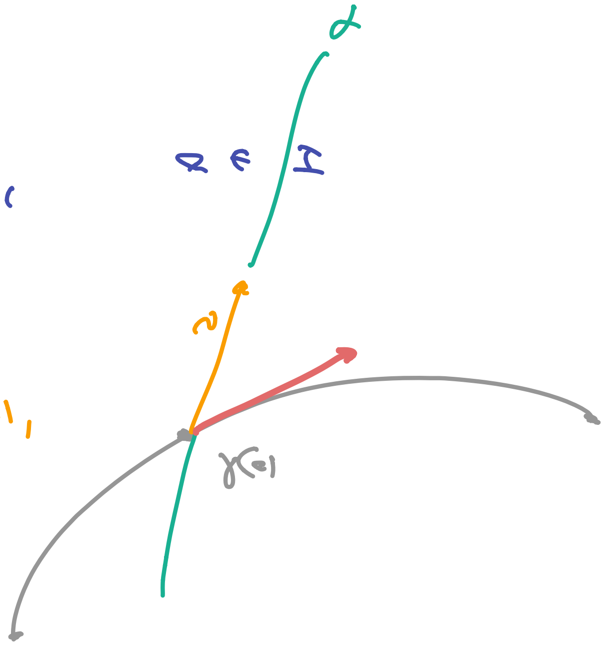
$\gamma_n : [0,1] \to [0,1]^2$

$\gamma_n \Rightarrow \gamma : [0,1] \to [0,1]^2$

Bsp.: [illegible] [illegible] [illegible]

Differenziation Kurven

$\gamma : I \to E$, $\quad a \in I$



$$\alpha(t) = \gamma(a) + v\,(t-a), \quad v \in E$$

($v = 0$ erlaubt)

$$\lim_{t \to a} \frac{\| \gamma(t) - \alpha(t) \|_E}{(t-a)} = 0$$

Beweis: (i) $\Leftrightarrow$ (ii)

Setze

$$z(t) = \underbrace{\frac{\gamma(t) - \gamma(a) - v\,(t-a)}{t-a}}_{\text{enthalten in } E}\,, \quad t \neq a$$

$$\xrightarrow[t\to a]{} 0$$

(ii) $\Leftrightarrow$ (iii)

$$\frac{\gamma(t) - \gamma(a)}{t-a} - v = z(t)\,, \quad t \neq a.$$

Wenn $z(t) \to 0$, $t \to a$, :

$$\lim_{t\to a} \frac{\gamma(t) - \gamma(a)}{t-a} = v$$

Bsp:

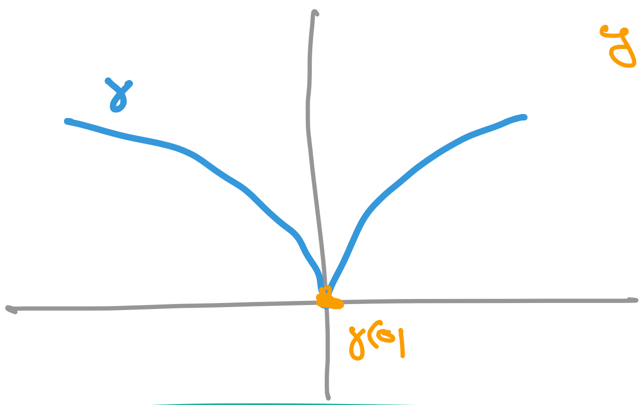
1. $\gamma : [-1,1] \to \mathbb{R}^2$

$\gamma(t) = (t^3, t^2)$

$x = t^3, \quad y = t^2$

Neilsche Parabel:

$y = x^{2/3}$

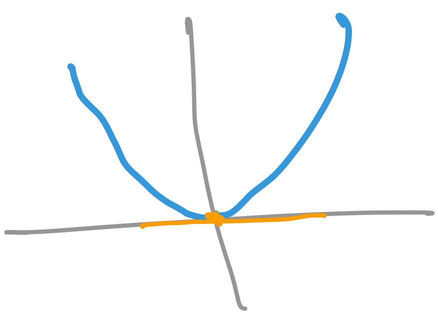


$\dot\gamma(t) = (3t^2, 2t) = 0$ für $t = 0$

$\|\dot\gamma(t)\|_2 = \sqrt{9t^4 + 4t^2}$

2. $\gamma : [-1,1] \to \mathbb{R}^2,$

$\gamma(t) = (t^3, t^6)$ $\quad y = x^2$



$\dot\gamma(t) = (3t^2, 6t^5)$

$= 0$ für $t = 0$