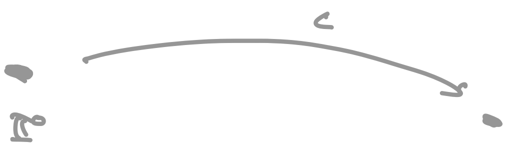
19. Vorlesung 22.12.21

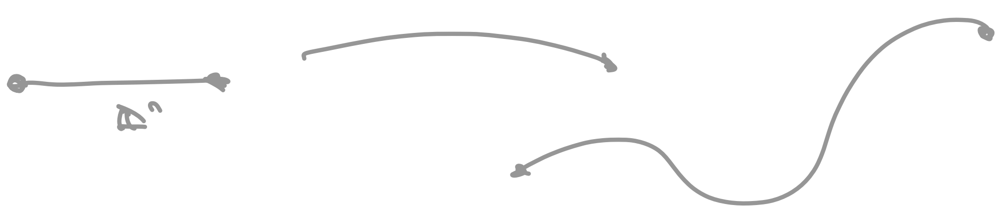
Bsp: 1. Ein 0-Würfel

$$c : \{0\} \longrightarrow U \subset \mathbb{R}^m$$



2. Ein 1-Würfel:

$$c : [0,1] \longrightarrow U \subset \mathbb{R}^m$$



3. Jeder 1-Würfel c ist eine 1-Kette: $1 \cdot c$

Beweis: Betrachte erst anschaulich.

j,β – Seite von i,α-Seite von I^n: $i \le j$

$$(I^n_{i,\alpha})_{j,\beta} : I^{n-2} \to \mathbb{R}^n$$

$$(I^n_{i,\alpha})_{j,\beta}(x) = I^n_{i,\alpha}(I^{n-1}_{j,\beta}(x))$$

$$= I^n_{i,\alpha}(x_1, \dots, \beta_j, \dots, x_n)$$

$$= (x_1, \dots, \alpha_i, \dots, \beta_{j+1}, \dots, x)$$

$$(I^n_{j+1,\beta})_{i,\alpha}(x) = I^n_{j+1,\beta}(I^{n-1}_{i,\alpha}(x))$$

$$= I^n_{j+1,\beta}(x_1, \dots, \alpha_i, \dots, x_n)$$

$$= (x_1, \dots, \alpha_i, \dots, \beta_{j+1}, \dots, x_n)$$

Also:
$$(I^n_{i,\alpha})_{j,\beta} = (I^n_{j+1,\beta})_{i,\alpha}$$

Also gilt

$$\boxed{(c_{i,\alpha})_{j,\beta} = (c_{j+1,\beta})_{i,\alpha}, \quad i \le j.}$$

Daraus folgt:

$$\partial(\partial c) = \partial\left(\sum_{i=1}^{n}\sum_{\alpha=0}^{1}(-1)^{i+\alpha} c_{i,\alpha}\right)$$

$$= \sum_{i,j=1}^{n}\sum_{\alpha,\beta=0}^{1}(-1)^{i+\alpha+j+\beta}(c_{i,\alpha})_{j,\beta}.$$

$$= 0.$$

Bsp: 1. Standard I^n

$\omega = f(x)\, dx_1 \wedge \dots \wedge dx_n$.

$$\int_{I^n}\omega = \int_{I^n} f(x)\, dx_1 \wedge \dots \wedge dx_n$$

$$= \int_{I^n} (I^n)^*\left(f \quad \dots \quad\right)$$

$$= \int_{I^n} f(x)\, dx_1 \wedge \dots \wedge dx_n = \int_{I^n} f\, dx.$$

2. $n = 1$:

$$\alpha = \sum_{i=1}^{n} \alpha_i(x)\, dx_i \quad \text{auf } U$$

$$c : I \longrightarrow U$$

$$\int_c \alpha = \int_I c^* \alpha = $$

$$= \int_I \sum_{i=1}^{n} \alpha_i(c(t)) \cdot \dot{c}_i(t)\, dt$$

$$= \int_I \langle \alpha \circ c , \dot{c} \rangle \, dt .$$

3. Sei $f : [a, b] \to \mathbb{R}$

$$c : I \longrightarrow [a, b]$$

$$\int_c f\, dx = \int_I c^* (f(x)\, dx)$$

$$= \int_0^1 (f \circ c)\, c' \cdot dt$$

$$= \int_{c(0)}^{c(1)} f\, dx$$

Substitution.

Beweis: Zuerst sei eine $(n-1)$-Form auf I^n:

$$\omega = \sum_{\mu=1}^{n} f_\mu(x)\, \underbrace{dx_1 \wedge \dots \wedge \widehat{dx_\mu} \wedge \dots \wedge dx_n}_{\omega_\mu}$$

auslassen!

Es genügt:

$$\omega = f\,\omega_\mu$$

$$\int_{I^n_{i,\alpha}} f\,\omega_\mu = \int_{I^n_{i,\alpha}} f\, dx_1 \wedge \dots \wedge \widehat{dx_\mu} \wedge \dots \wedge dx_n$$

$$= 0, \quad i \neq \mu.$$

$i = \mu$:

$$\int_{I^n_{\mu,\alpha}} f\,\omega_\mu = \int_{I^{n-1}} (I^n_{\mu,\alpha})^*(f\,\omega_\mu)$$

$$= \int_{I^{n-1}} f(\dots, \alpha_{\mu}, \dots)\, dx^{n-1}$$

Daraus folgt:

$$\int_{\partial I^n} f\,\omega_\mu$$

$$= \sum_{i=1}^{n} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \int_{I^n_{i,\alpha}} f\,\omega_\mu$$

$$\underset{i=\mu}{=} (-1)^{\mu+1} \int_{I^n_{\mu,1}} f\,\omega_\mu + (-1)^{\mu} \int_{I^n_{\mu,0}} f\,\omega_\mu$$

$$= (-1)^{\mu+1} \int_{I^{n-1}} \underbrace{\left\{ f(\ldots,1_\mu,\ldots) - f(\ldots,0_\mu,\ldots) \right\}}_{\int_{I^1} \partial_\mu f \, dx_\mu} d^{n-1}x$$

μ-te Koordinate

$$\underset{\text{Fubini}}{=} (-1)^{\mu+1} \int_{I^n} \partial_\mu f \, d^n x$$

$$= (-1)^{\mu+1} \int_{I^n} \partial_\mu f \, dx_1 \wedge \ldots \wedge dx_n ,$$

Dann gilt:

$$d(f\omega_\mu)$$

$$= df \wedge dx_1 \wedge \dots \wedge \widehat{dx_\mu} \wedge \dots \wedge dx_n$$

$$\left(\sum f_k \, dx_k\right)$$

$$= f_\mu \, dx_\mu \wedge \left(dx_1 \wedge \dots \wedge \widehat{dx_\mu} \wedge \dots \wedge dx_n\right)$$

$$= (-1)^{\mu-1} f_\mu \, dx_1 \wedge \dots \wedge dx_n .$$

Also Stokes ist gezeigt:

$$\int_{\partial I^n} f\omega_\mu = (-1)^{\mu+1} \int_{I^n} \partial_\mu f \, dx_1 \wedge \dots \wedge dx_n$$

$$= \int_{I^n} d(f\omega_\mu) .$$

Allgemeine Würfel:

$$\int_c d\omega = \int_{I^n} c^*(d\omega)$$

$$= \int_{I^n} d(c^*\omega)$$

$$\underset{\text{Spezieller Fall}}{=} \int_{\partial I^n} c^*\omega$$

$$= \sum_{i=1}^{n} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \int_{I^n_{i,\alpha}} c^*\omega$$

$$\int_{I^n_{i,\alpha}} c^*\omega = \int_{I^{n-1}} (I^n_{i,\alpha})^*(c^*\omega)$$

$$= \int_{I^{n-1}} (c\circ I^n_{i,\alpha})^*\omega$$

$$\underset{\text{Def.}}{=} \int_{I^{n-1}} c_{i,\alpha}^*\omega$$

$$= \int_{c_{i,\alpha}} \omega$$

Also:

$$\int_c d\omega = \sum_{i=1}^{n} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \int_{c_{i,\alpha}} \omega = \int_{\partial c} \omega .$$

Für eine Kette $c = \lambda_1 c_1 + \dots + \lambda_r c_r$:

$$\int_c d\omega = \sum_{1 \le i \le r} \lambda_i \int_{c_i} d\omega$$

$$= \sum \lambda_i \int_{\partial c_i} \omega$$

$$= \int_{\partial c} \omega .$$