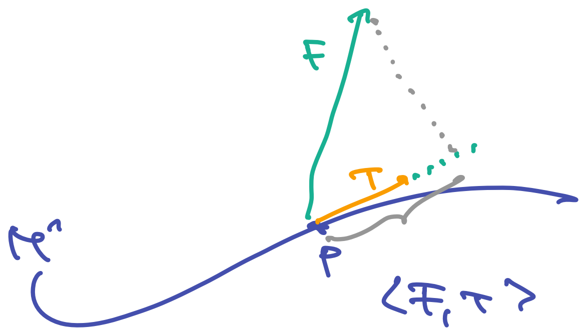
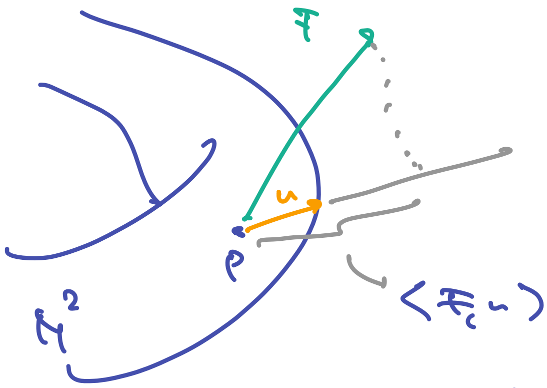
24. Vorlesung 26.1.2022



$\langle F, T \rangle$ skalare Funktion auf M^1

ds skalares Linienelement



$\langle F, n \rangle$ skalare Funktion auf M^2

dA skalares Flächenelement

Beweis:

$$\varphi^* dA = f(u,v)\, du \wedge dv$$

und

$$f(u,v) = \varphi^* dA(e_1, e_2)$$

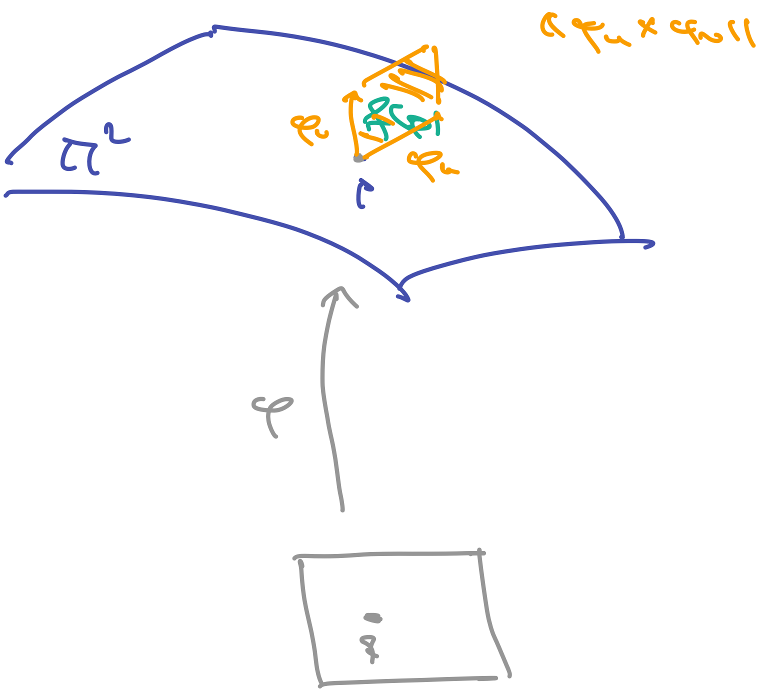
$$= dA(\varphi_* e_1, \varphi_* e_2)$$

$$= dA(\varphi_u, \varphi_v)$$

$$= \|\varphi_u \times \varphi_v\|$$

LA:

$$\|v \times w\|^2 = \|v\|^2 \|w\|^2 - \langle v, w\rangle^2 .$$



Beispiel 1: $f: \mathbb{R}^2 \to \mathbb{R}$

Graph von f: Γ 2-dim. in $\mathbb{R}^3$

Koordinatensystem:

$$\varphi: \mathbb{R}^2 \to \mathbb{R}^3$$
$$(u,v) \mapsto (u, v, f(u,v))^T$$

Dann:

$$\varphi_u = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix}, \quad \varphi_v = \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix}$$

$$\varphi_u \times \varphi_v = \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix}$$

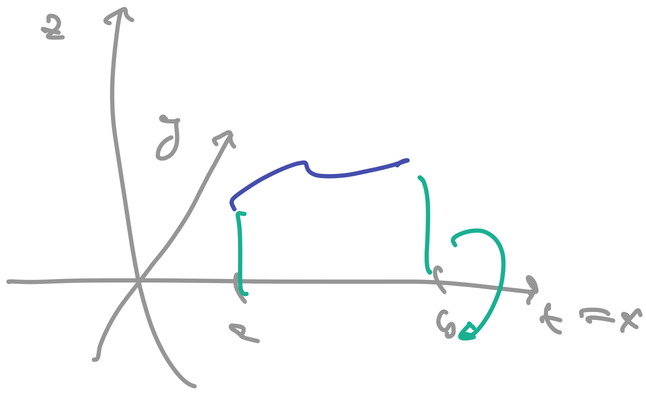
$$\| \varphi_u \times \varphi_v \|^2 = 1 + f_u^2 + f_v^2 .$$

Flächeninhalt von Γ:

$$|\Gamma| = \int_\Gamma dA = \int_D \sqrt{1 + f_u^2 + f_v^2} \, d\lambda_2 .$$

$$\int_\Gamma \psi \, dA = \int_D \psi(u, v, f(u,v)) \cdot$$

Beispiel 2: $\quad f : [a,b] \to (0,\infty)$



Rotationsfläche:

$$\varphi : [a,b] \times [0,2\pi) \longrightarrow \mathbb{R}^3$$

$$\varphi(t,\theta) = \begin{pmatrix} t \\ f(t) \cos\theta \\ f(t) \sin\theta \end{pmatrix}$$

Dann:

$$\varphi_t = \begin{pmatrix} 1 \\ f' \cos\theta \\ f' \sin\theta \end{pmatrix}, \quad \varphi_\theta = \begin{pmatrix} 0 \\ -f \sin\theta \\ f \cos\theta \end{pmatrix}$$

Somit

$$E = \|\varphi_t\|^2 = 1 + (f')^2$$

$$G = \|\varphi_\theta\|^2 = f^2$$

$$F = \langle \varphi_t, \varphi_\theta \rangle = 0.$$

Also:

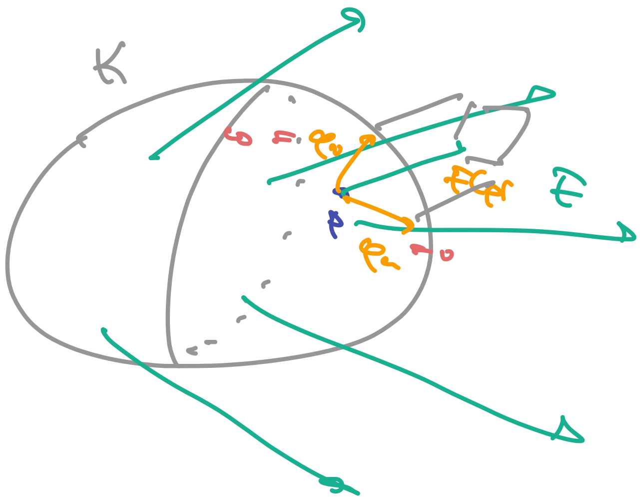
$$\varphi^* dA = \underbrace{f}_{\geq 0} \cdot \sqrt{1 + f'^2} \; dt \wedge d\theta$$

$$|R| = \int_R dA =$$

$$= \int_0^{2\pi} \int_a^b$$

$$= 2\pi \int_a^b f(x) \cdot \sqrt{1 + (f'(x))^2} \, dx .$$

$\blacksquare$



$$F \cdot (v \times w) = F \cdot (\varphi_u \times \varphi_v)$$

$$= F \cdot d\vec{A}(v, w)$$

Umformt:

$$\underbrace{\int_{\partial K} F \cdot d\vec{A}}_{\text{Strömungsdichte}} = \int_K \operatorname{div} F \cdot dV.$$

$$K = B_r(x), \qquad r \to 0$$

$$\operatorname{div} F(x) = \lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \operatorname{div} F \cdot dV$$

$$= \lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{\partial B_r(x)} F \cdot d\vec{A}$$

infinitesimale Strömungsdichte.

Beweis:

$$\operatorname{div} F = 0$$

$$\Longrightarrow \quad d\big(\underbrace{F \cdot d\vec{A}}_{\text{exakt}}\big) = 0$$

$\longrightarrow$ auf sternförmigem Gebiet : geschlossen:

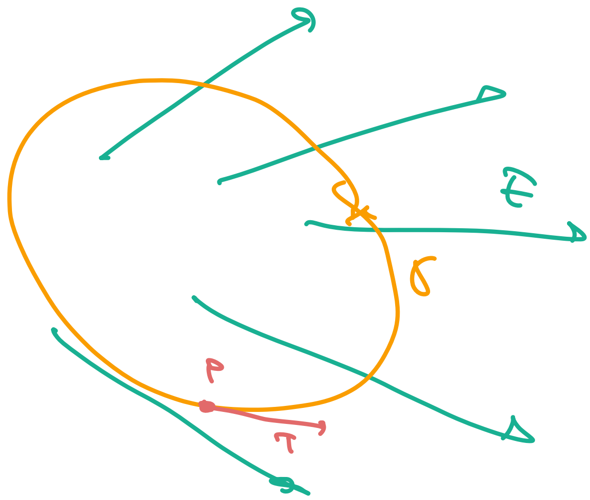
d. h. es gibt $U \cdot d\vec{s}$:

$$F \cdot d\vec{A} = d\big(U \cdot d\vec{s}\big) = \operatorname{rot} U \cdot d\vec{A}$$

Also:

$$F = \operatorname{rot} U.$$

$\blacksquare$



$$\int_\gamma F \cdot d\vec{s} = \int_\gamma \langle F, T \rangle \, ds$$

$\gamma = \partial M$:

$$\int_{\partial M^2} F \cdot d\vec{s} = \int_{M^2} \underbrace{\operatorname{rot} F}_{\text{infinitesimale Verwirbelung}} \cdot d\vec{A}$$

Dem.:

$$\operatorname{rot} F = 0$$

$$\Rightarrow \quad \int \operatorname{rot} F \cdot d\vec{A} = 0$$

$$\Rightarrow \quad \underbrace{d(F \cdot d\vec{s})}_{\text{exact}} = 0$$

so gradient:

$$F \cdot d\vec{s} = df = \nabla f \cdot d\vec{s}.$$

So: $\quad F = \nabla f.$

Base: ∂B_0 $r = 0$.

Define

$$\varphi_r(x) := f(rx), \qquad r \geq 0$$

Define

$$I(r) = \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} \varphi_r \, dA.$$

Dem:

$$I(0) = \frac{1}{|\ \ |} \int_{\partial B_1(0)} f(0) \, dA = f(0)$$

Goal: $I(r)$ constant in r.

So: $I(r) = f(0)$

z.z.: $I'(r) = 0$:

$$\frac{d}{dr} I(r) = \frac{1}{|\cdot|} \int_{\partial B_1(0)} \frac{d}{dr} f(rx) \, dA$$

$$= \frac{1}{|\cdot|} \int_{\partial B_1(0)} \nabla f(rx) \cdot \underset{\substack{\| \\ r \cdot n}}{x} \, dA \qquad \Longleftrightarrow \quad n = \frac{x}{r}.$$

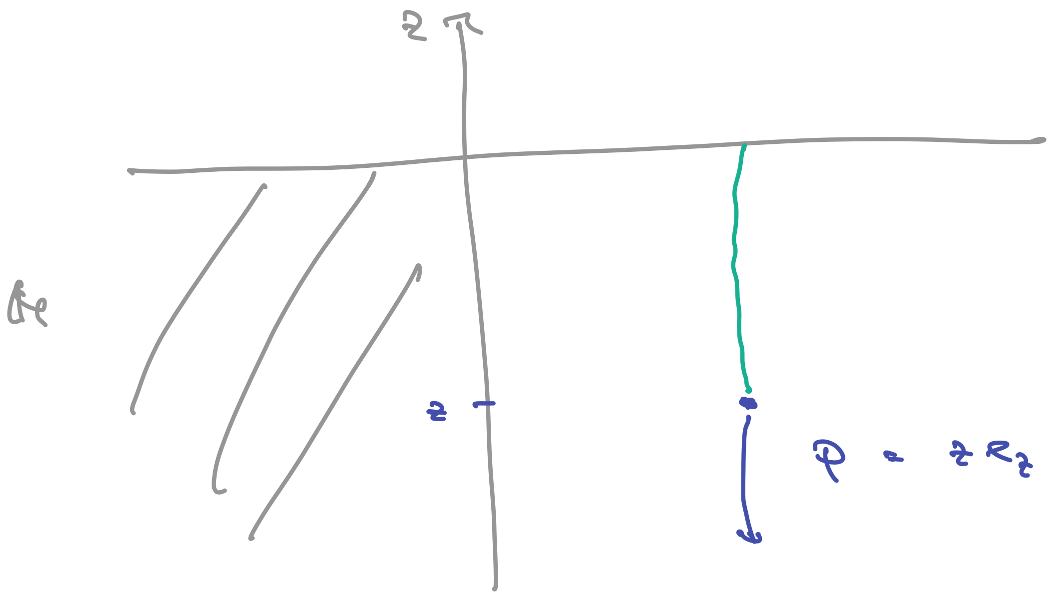
$$= \frac{1}{|\cdot|} \, r \int_{\partial B_r(0)} \nabla f(rx) \cdot n \, dA$$

$$= \frac{1}{|\cdot|} \, r^k \int_{B_r(0)} \underbrace{\Delta f(rx)}_{=0} \, dV$$

$$= 0.$$

$\square$

Beispiel: $\Omega = \{ z \leq 0 \}$ $\quad P = z$



$$P_n = \langle P, n \rangle \, n$$

Komponente

$$F = \int_{\partial\Omega} \langle P, n \rangle \, dA$$

$P = z e_z$

$\operatorname{div} P = 1$

Divergenzsatz:

$$= \int_\Omega \operatorname{div} P \cdot dV$$

$$= \int_\Omega dV = |\Omega| .$$