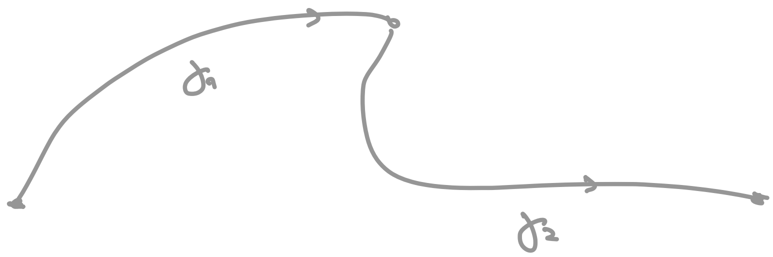
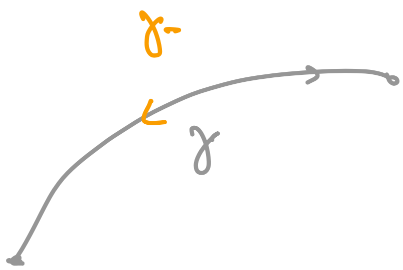
7. Vorlesung 10.11.2021

Wegadditivität



$\gamma = \gamma_1 + \gamma_2$

$$\int_\gamma \alpha = \int_{\gamma_1+\gamma_2} \alpha = \int_{\gamma_1} \alpha + \int_{\gamma_2} \alpha$$



$$\int_{\gamma^-} \alpha = -\int_\gamma \alpha \ .$$

Beweis:

$$\left| \int_\omega \alpha \right| = \left| \int_a^b \alpha(\gamma(t)) \cdot \gamma'(t)\, dt \right|$$

$$\leq \int_a^b | \underbrace{\alpha(\gamma(t))\, \gamma'(t)}_{\in \mathbb{R}} |\, dt$$

$$\leq \int_a^b \underbrace{\| \alpha(\gamma(t)) \|_{V^*}}_{\text{stetig in } t} \| \gamma'(t) \|_V \, dt$$

$$\leq \max_{p \in \omega} \| \alpha(p) \|_{V^*} \underbrace{\int_a^b \| \gamma'(t) \| \, dt}_{L(\omega)} . \quad \blacksquare$$

Bsp: 1. $I = (a,b)$

$\alpha = a(x)\, dx$, a stetig auf (a,b)

Setze:

$$f(x) = \int_{x_0}^{x} a(t)\, dt$$

f ist C^1:

$$df(x) = f'(x)\, dx = a(x) dx.$$

Dann α ist auf (a,b) ist exakt.

2. Auf $\mathbb{R}^n \setminus \{0\}$: $\varphi : (0,\infty) \to \mathbb{R}$ stetig

$$\alpha = \varphi(\|x\|) \sum_{k=1}^{n} x_k \, dx_k$$

$$= \varphi(\|x\|) \, \langle x, \cdot \rangle$$

$$= \varphi(\|x\|) \, v_x$$

Setze:

$$f(x) = F(\|x\|) = \int_1^{\|x\|} t\varphi(t)\,dt .$$

Dann: $\|x\| = \sqrt{\sum x_i^2}$

$$d(\|x\|) = \sum \frac{x_k}{\|x\|} dx_k \, , \quad x \neq 0$$

Also:

$$df(x) = F'(\|x\|) \, d(\|x\|)$$

$$= \|x\| \cdot \varphi(\|x\|) \cdot \sum_{k=1}^{n} \frac{x_k}{\|x\|} dx_k$$

$$= \varphi(\|x\|) \cdot \sum_{k=1}^{n} x_k \, dx_k = \alpha .$$

$\square$

Beweis: Sei ω exakt.

Sei $\gamma : [a,b] \to U$ C^1-Param.

Sei $\alpha = df$

Dann ist $f \circ \gamma$ C^1 !

$$\underbrace{(f\circ\gamma)'(t)} = \underbrace{\cancel{Df}(\gamma(t))} \cdot \dot\gamma(t)$$

$$= df(\gamma(t)) \cdot \dot\gamma(t)$$

$$= \alpha(\gamma(t))\, \dot\gamma(t).$$

Also:

$$\boxed{\int_\omega \alpha =} \int_\gamma df = \int_a^b \underbrace{df(\gamma(t))\,\dot\gamma(t)}\, dt$$

$$= \int_a^b (f\circ\gamma)'(t)\, dt$$

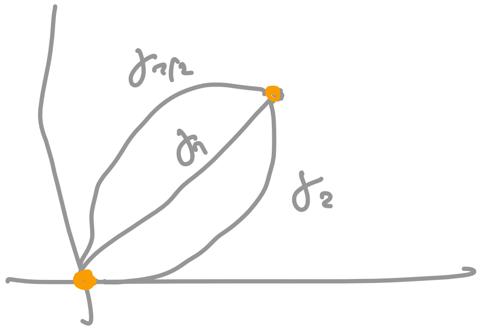
$$= (f\circ\gamma)\Big|_a^b$$

$$= f\Big|_{\gamma(a)}^{\gamma(b)}$$

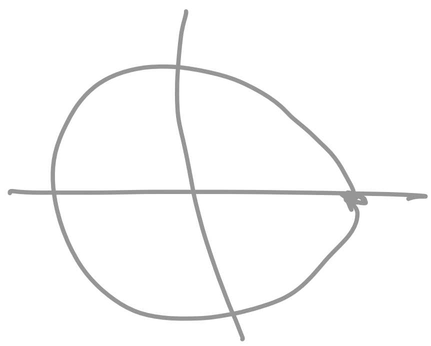
$$\boxed{= f\Big|_{\omega_a}^{\omega_b}}.$$

$\blacksquare$

$$\alpha = y^2\,dx + dy$$

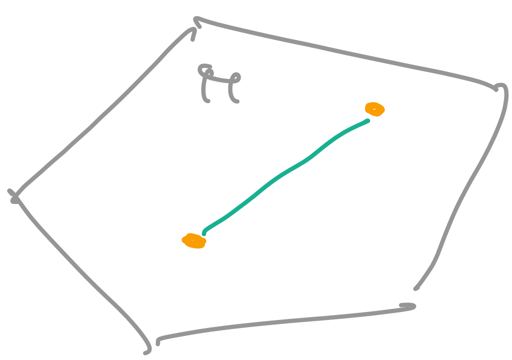


$$v = \frac{x\,dy - y\,dx}{x^2+y^2}$$ nicht exakt.



Bsp: 1. Jedes offene Intervall $I \subset \mathbb{R}$.

2. Jede konvexe Menge in $\mathbb{C}$ ist [illegible].



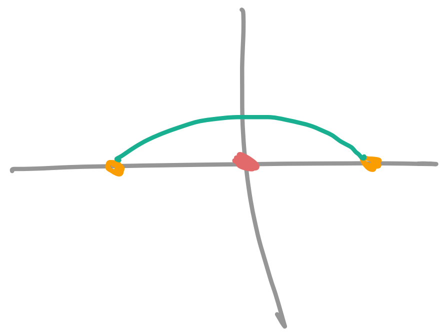
3. $\mathbb{R}^n \setminus \{0\}$

$n=1$

nicht zshgd.

$n \geq 2$

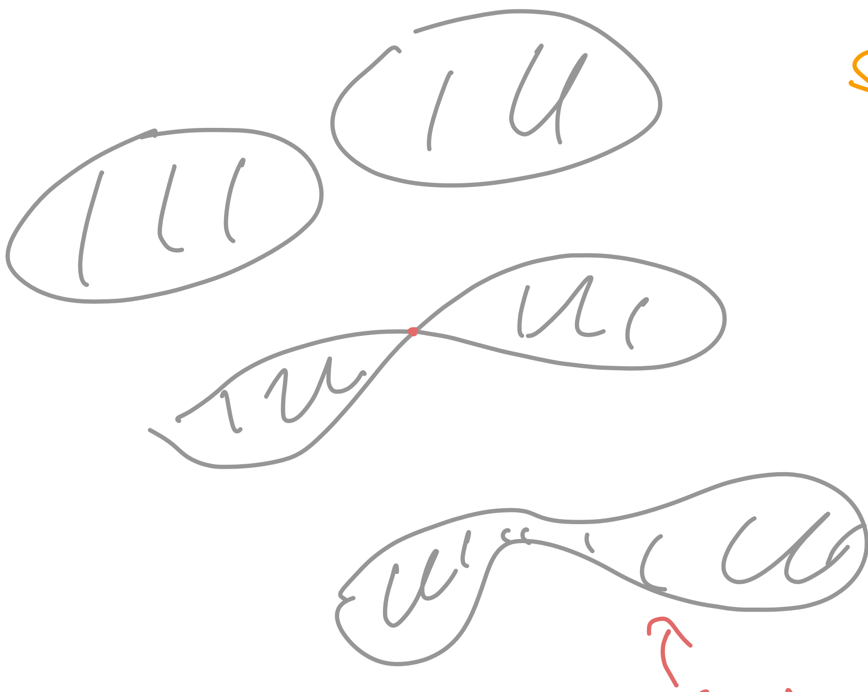
wegzshgd.



Satz offen

nicht

wegzshgd.



wegzshgd.

Beweis: $\Rightarrow$ trivial ✓

$\Leftarrow$ Fixiere $x_0 \in \Omega$. Sei $x \in \Omega$ beliebig.

Dann ex. D^1-Kurve

$$\gamma : [a,b] \to \Omega,$$

$$\gamma(a) = x_0, \quad \gamma(b) = x.$$

Dann $g = f \circ \gamma$ D^1.

Da $Df \equiv 0$ auf Ω:

$$g'(t) = \underbrace{Df(\gamma(t))}_{=0} \cdot \gamma'(t) = 0, \quad t \in (a,b)$$

Somit ist g sogar C^1, und wegen $g' = 0$ auch konstant.

Also:

$$g(a) = g(b)$$

$$\Longleftrightarrow \quad \underbrace{f(x_0)}_{=\text{fest}} = \underbrace{f(x)}_{\text{beliebig in } \Omega}.$$

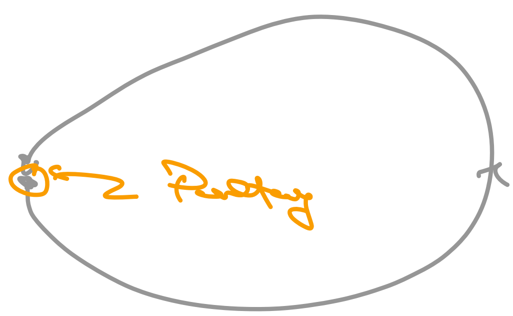
$\Rightarrow$ f ist konstant auf Ω $\square$.

Beispiele für Gebiete:

1. Nichtleere offene Intervalle
2. Nichtleere offene konvexe Teilmenge $\subset \mathbb{C}$
3. $\Omega_\varepsilon = \{(u,v) : v^2 \geq u^2 - \varepsilon\}$ für $\varepsilon > 0$, aber nicht für $\varepsilon \leq 0$
4. Vereinigung $\Omega_1 \cup \Omega_2$ von Gebiete, falls $\Omega_1 \cap \Omega_2 \neq \emptyset$.

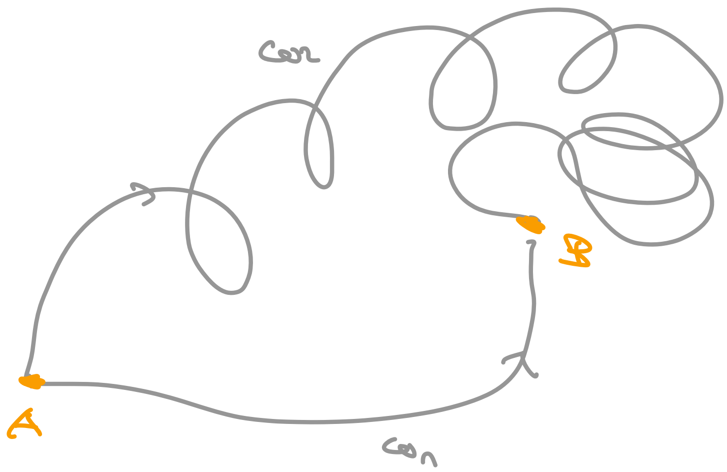
Beweis: (i) $\Rightarrow$ (ii) trivial.

(ii) $\Rightarrow$ (iii)



$$\int \alpha = 0$$

$(iii) \Longrightarrow (ii)$



$\omega_1 - \omega_2$ geschlossen

$$\Longrightarrow \int_{\omega_1 - \omega_2} \alpha = 0$$

$$= \int_{\omega_1} \alpha + \int_{-\omega_2} \alpha = \int_{\omega_1} \alpha - \int_{\omega_2} \alpha$$

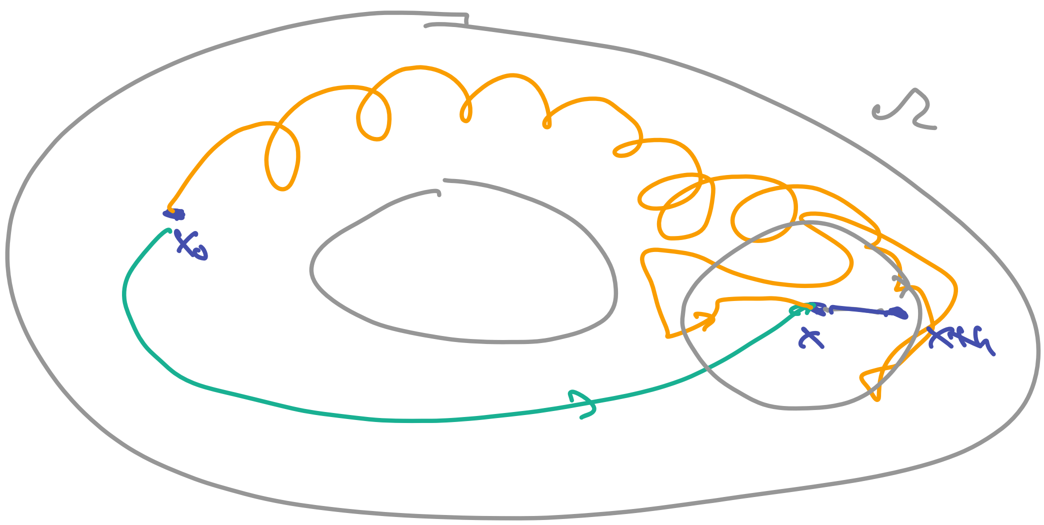
$$\Longrightarrow \int_{\omega_1} \alpha = \int_{\omega_2} \alpha .$$

Cil $\longrightarrow$ Cil   Dr $\int \alpha$ [illegible] :

Defin $f : \Omega \longrightarrow \mathbb{R}$

$$f(x) = \int_{x_0}^{x} \alpha \quad , \quad x \in \Omega$$

[illegible] $x_0 \in \Omega$ [illegible] [illegible].   [illegible]



$$f(x+h) - f(x) = \int_{x_0}^{x+h} \alpha - \int_{x_0}^{x} \alpha$$

$$= \int_{[x,\, x+h]} \alpha$$

Param: $x(t) = x + th, \quad 0 \le t \le 1$

$$= \int_0^1 \alpha(x+th)\, h \, dt$$

Subtrahiere $\alpha(x) h$:

$$f(x+h) - f(x) - \alpha(x) h$$

$$= \underbrace{\int_0^1 [\underbrace{\alpha(x+th) - \alpha(x)}_{\xrightarrow{h \to 0} 0}]\, h \, dt}_{o(h)}$$

Also:

$$\boxed{f(x+h) = f(x) + \alpha(x) h + o(h).}$$

Also: f ist in Punkt x diff'bar:

$$Df(x) h = \alpha(x) h.$$

$$= df(x) h \qquad \text{gilt für alle } h$$

$$\Longrightarrow \quad df = \alpha.$$

$$\frac{\partial a_k}{\partial x_\ell} = \frac{\partial a_\ell}{\partial x_k}\,, \qquad 1 \le k, \ell \le n$$

Proof: Assume:

$$\alpha = df = \sum_{k=1}^{n} \partial_k f \, dx_k\,,$$

where $f \in C^1$.

But α and C^1, so that $f \in C^2$:

Lemma von Schwarz:

$$\underbrace{(f_{x_k})_{x_\ell}}_{a_k} = f_{x_k x_\ell} = f_{x_\ell x_k} = \underbrace{(f_{x_\ell})_{x_k}}_{a_\ell}$$

$$\frac{\partial a_k}{\partial x_\ell} = \frac{\partial a_\ell}{\partial x_k}$$

Bsp.: 1. Auf $\mathbb{R}^2$ ist

$$u\,dx + v\,dy$$

geschlossen genau wenn

$$\boxed{\begin{aligned} u_y &= v_x \\ \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x}. \end{aligned}}$$

2. Für $\alpha = y^2\,dx + dy$:

nicht geschlossen:

$$\left.\begin{aligned} (y^2)_y &= 2y \\ (1)_x &= 0 \end{aligned}\right\} \neq$$

3. Winkelform:

$$\omega = \frac{y\,dx - x\,dy}{x^2+y^2}$$ geschlossen.

$$\left.\begin{aligned} &\partial_y\left(\frac{y}{x^2+y^2}\right) \\ &\partial_x\left(\frac{-x}{x^2+y^2}\right) \end{aligned}\right\} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \checkmark$$

4. $\mathbb{R}^3$:

$$\alpha = u\,dx + v\,dy + w\,dz$$

IB:

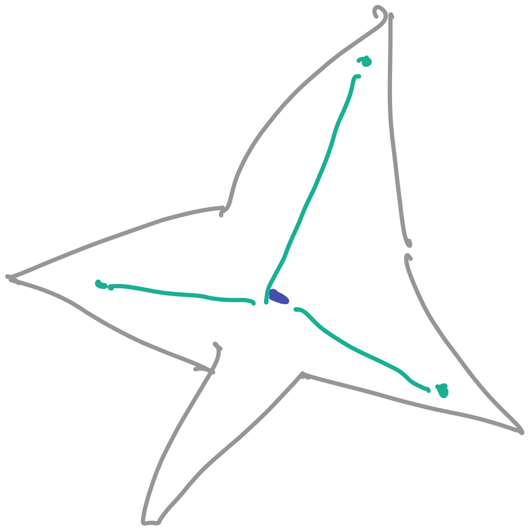
$$\begin{cases} u_y = v_x \\ u_z = w_x \\ v_z = w_y \end{cases}$$

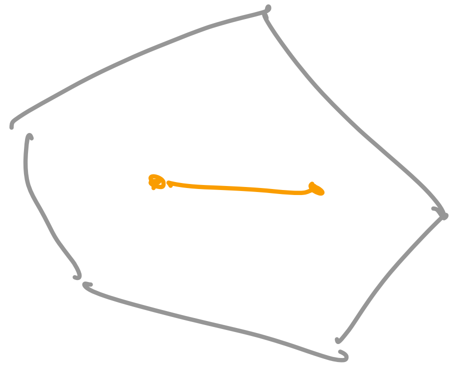
Äquivalent:

$$\nabla \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

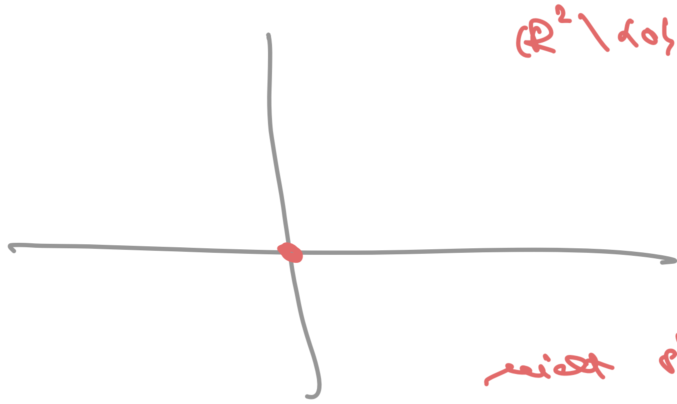
$$= \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix} = 0 .$$

$\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ hat <u>Rotation</u> 0.





$\mathbb{R}^2 \setminus \{0\}$



nicht sternförmig

Voraussetzung: Sei $\alpha = df$,

also:

$$\alpha(x) = \sum a_k(x)\, dx_k = \sum \partial_k f(x)\, dx_k.$$

Dann gilt auch:

$$\alpha(tx)\,x = \sum a_k(tx)\, x_k = \sum \partial_k f(tx)\, x_k$$

$$= \partial_t f(tx)$$

auf $(0,1)$ integrieren

(Idee = Integral)