8. Vorlesung

11.11.2021

Beweis: Sei α geschlossen und C^1, und sei $\Omega \subset \mathbb{R}^n$ sternförmig. OBdA sei $0 \in \Omega$ Zentrum für Ω.

Def.:

$$f(x) = \int_0^1 \alpha(tx)x \, dt = \int_0^1 \sum_{l=1}^n a_l(tx)\, x_l \, dt$$

wohldefiniert für $x \in \Omega$, da $[0,x] \subset \Omega$

Damit

$$f : \Omega \to \mathbb{R}.$$

Da a_l C^1-Funktion: f ist stetig diffbar, und

$$\partial_k f(x) = \int_0^1 \sum_{l=1}^n \partial_k \left(a_l(tx)\, x_l \right) dt$$

$$= \int_0^1 \Big(a_k(tx) + t \sum \underbrace{\partial_k a_l}_{= \partial_l a_k} (tx)\, x_l \Big) \, dt$$

(G.

Beweis

$$a_k(tx) + t \sum_{\ell=1}^{n} \partial_\ell a_k(tx)\, x_\ell \overset{!}{=} \partial_t \big(t\, a_k(tx)\big)$$

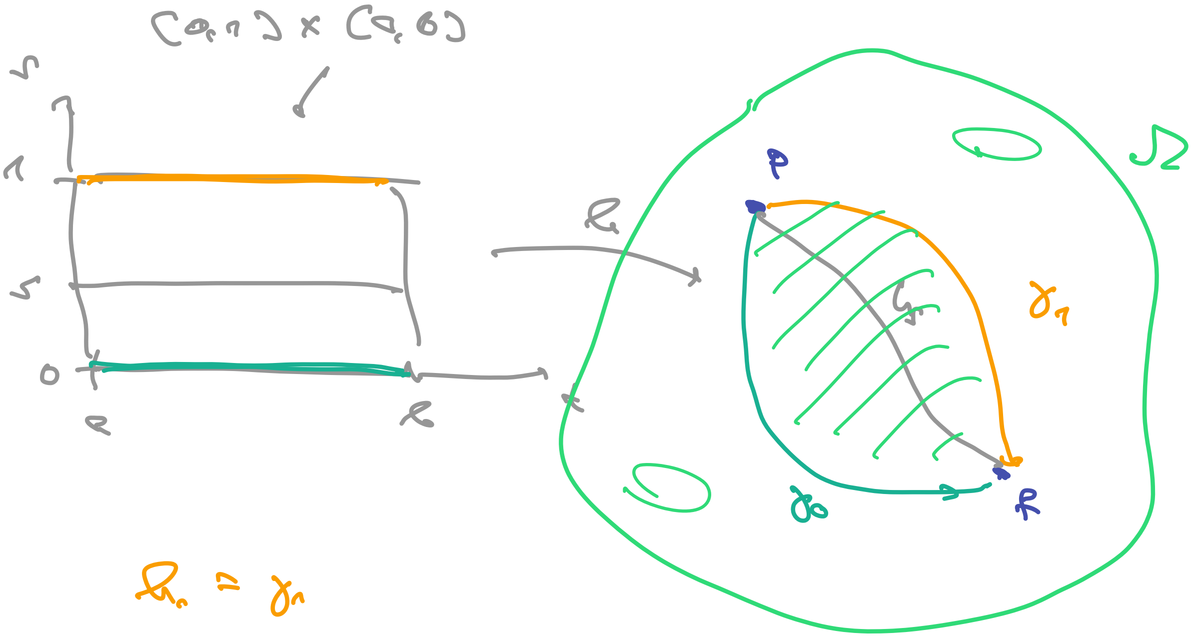
$$= \int_0^1 \partial_t \big(t\, a_k(tx)\big)\, dt$$

$$= t\, a_k(tx) \Big|_0^1$$

$$= a_k(x) \underset{\text{s.o.}}{=} \partial_k f(x).$$

Also gilt:

$$df = \alpha. \qquad \blacksquare$$



$H_1 = \gamma_1$

$H_0 = \gamma_0$

$H_s(a) = p \quad , \quad H_s(b) = q \, .$

!

$H([0,1] \times [a,b]) \subset \Omega \, .$

Bsp: $\gamma_0, \gamma_1 : [a, b] \longrightarrow \Omega$

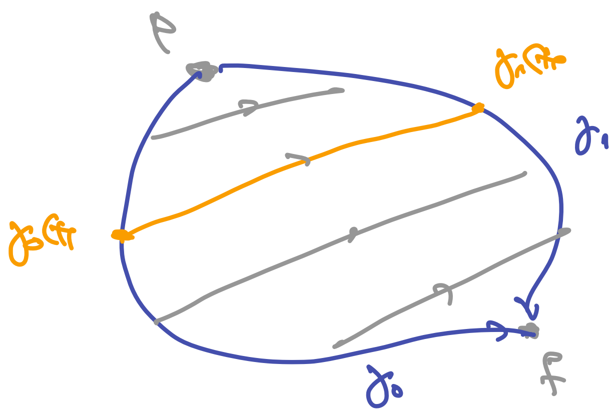
$$\begin{cases} \gamma_0(a) = \gamma_1(a) \\ \gamma_0(b) = \gamma_1(b) \end{cases}$$

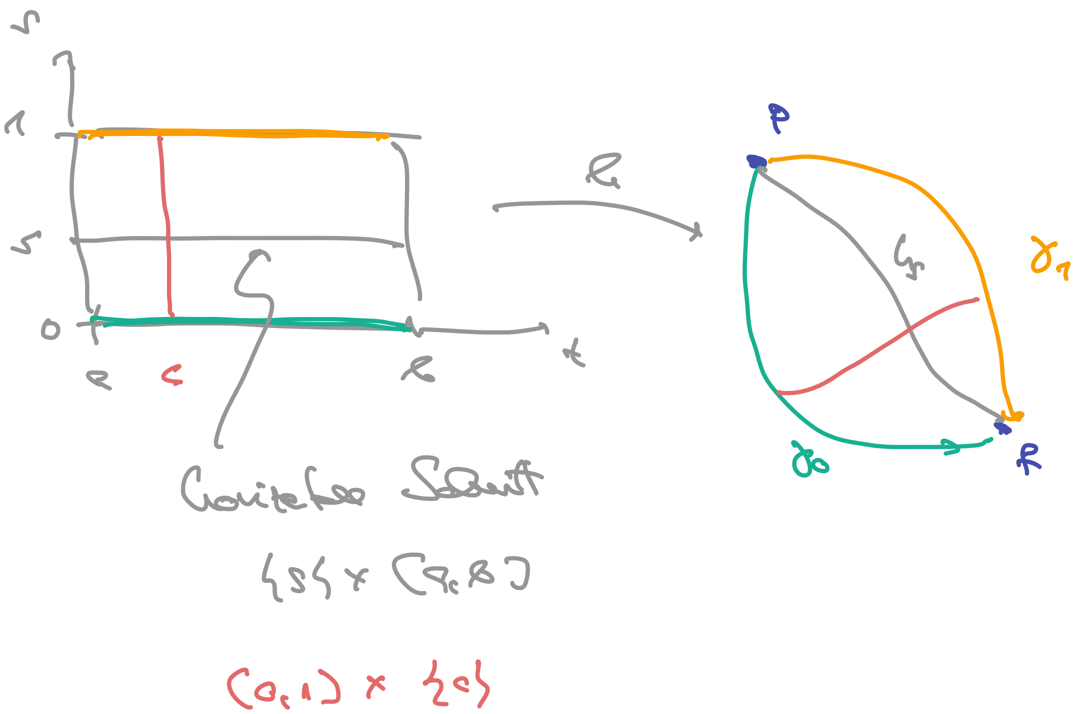
Wenn

$$[\gamma_0(t), \gamma_1(t)] \subset \Omega, \quad 0 \leq t \leq 1$$

Dann sind γ_0 und γ_1 homotop:

$$h(s,t) := (1-s)\,\gamma_0(t) + s\,\gamma_1(t).$$



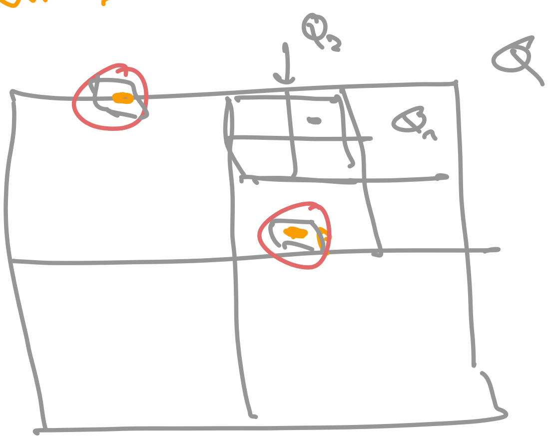

s
1
s
0
a
c
b
t
H
p
q
γ_s
γ_1
γ_0
$\{s\} \times [a,b]$
$(0,1) \times \{c\}$

Beweis: Sei $Q = [0,1] \times [0,1]$,

$f : Q \longrightarrow \Omega$ stetige Funktion

1. "Einfache Zerlegung":

"Zerlegung, dann weiter":

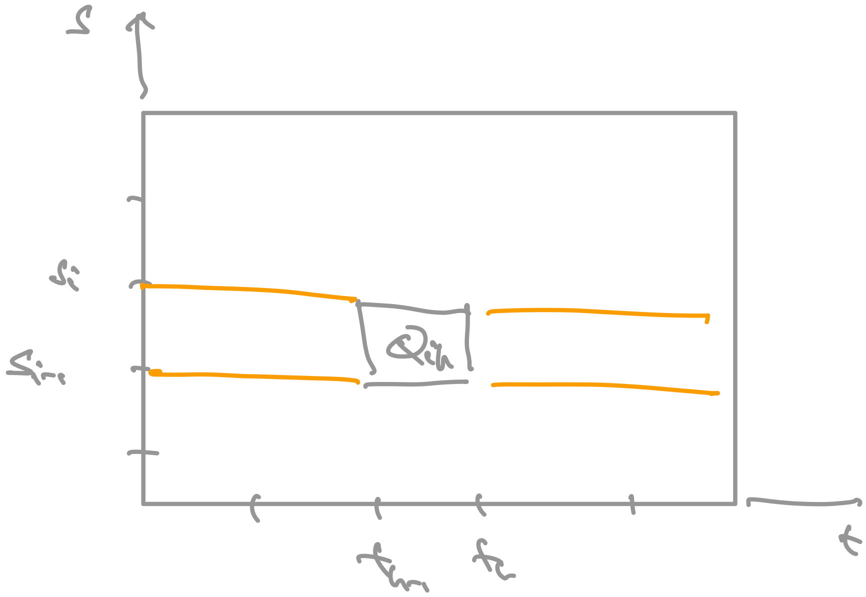


Gesucht: $Q \supset Q_1 \supset Q_2 \supset \dots$,

wo α auf $f(Q_j)$ keine Schranke.

$$\bigcap_{k \geq 0} Q_k = \{p\} \subset \Omega$$

Folgt wenn p ist [illegible] α [illegible].



$$Q_{ik} = (s_{i-1}, s_i] \times (t_{k-1}, t_k],$$

$1 \le i \le m$

$1 \le k \le n.$

α is [illegible] [illegible] [illegible] op

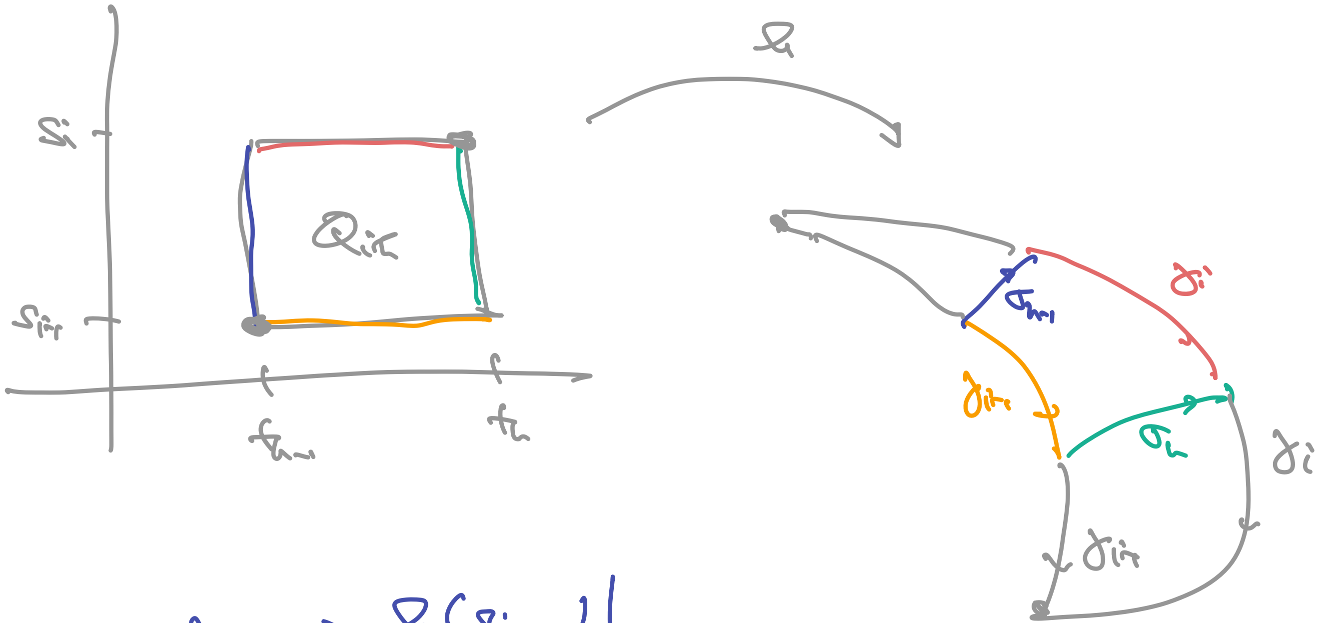
Q_{ik} [illegible].

2. Zeigen : $\displaystyle\int_{\gamma_{i-1}} \alpha = \int_{\gamma_i} \alpha,$

waar $\gamma_i = \Sigma(s_i, \cdot) = \Sigma_{s_i}$.

Daarom?

$$\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha.$$



$$\gamma_{i,k} = h(s_i, \cdot)\big|_{[t_{k-1}, t_k]}$$

$$\sigma_k = h(\cdot, t_k)\big|_{[s_{i-1}, s_i]}$$

$$\boxed{\gamma_{i-1,k} + \sigma_k = \sigma_{k-1} + \gamma_{i,k}}$$

Da α exakt auf Umgebung von Q_{ik}:

$$\boxed{\int_{\gamma_{i-1,k}} \alpha + \int_{\sigma_k} \alpha = \int_{\sigma_{k-1}} \alpha + \int_{\gamma_{i,k}} \alpha}$$

Dies gilt für alle k. Summe über k:

$$\int_{\gamma_{i-1}} \alpha + \sum_k \int_{\sigma_k} \alpha = \sum_k \int_{\sigma_{k-1}} \alpha + \int_{\gamma_i} \alpha$$

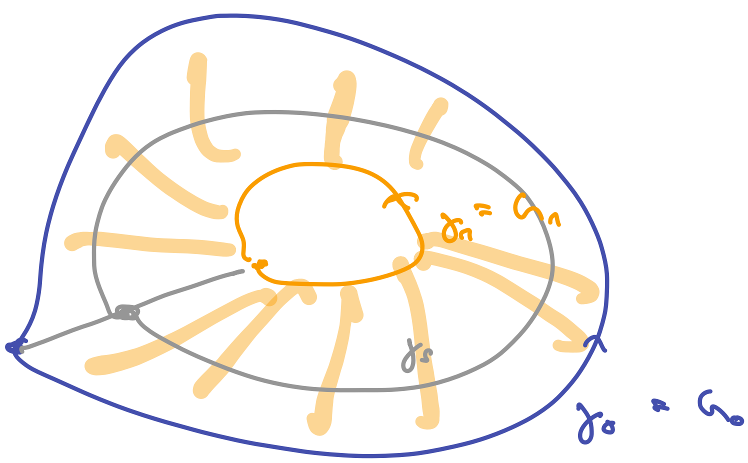
$$\Longrightarrow \int_{\gamma_{i+1}} \alpha + \underbrace{\cancel{\int_{\sigma_0} \alpha}}_{} = \cancel{\int_{\sigma_m} \alpha} + \int_{\gamma_i} \alpha .$$

0 0

Randstücke

$$\Longrightarrow \int_{\gamma_{i+1}} \alpha = \int_{\gamma_i} \alpha .$$

$\boxed{\text{QED}}$



$$\sigma : [0,1] \times [0,1] \longrightarrow \Omega$$

$$\gamma_s = \sigma(s, \cdot) = \sigma_s$$

Beweis für freie Homotopie:

Dies gilt für alle k. Summe über k:

$$\int_{\gamma_{i-1}} \alpha + \sum_k \int_{\sigma_k} \alpha = \sum_k \int_{\sigma_{k-1}} \alpha + \int_{\gamma_i} \alpha$$

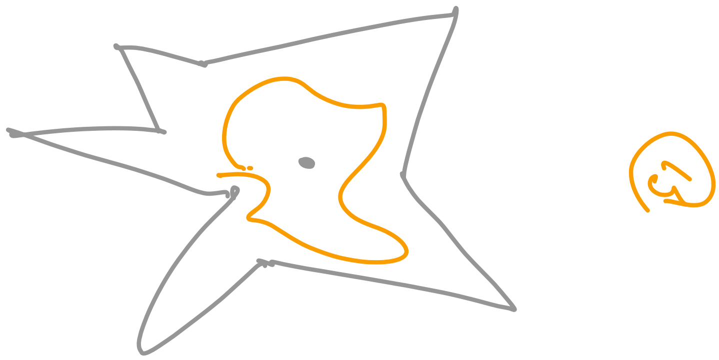
$$\Longrightarrow \boxed{\int_{\gamma_{i-1}} \alpha + \int_{\sigma_0} \alpha = \int_{\sigma_m} \alpha + \int_{\gamma_i} \alpha .}$$

$$\Longrightarrow \int_{\gamma_{i-1}} \alpha = \int_{\gamma_i} \alpha .$$

∎

Beispiele:

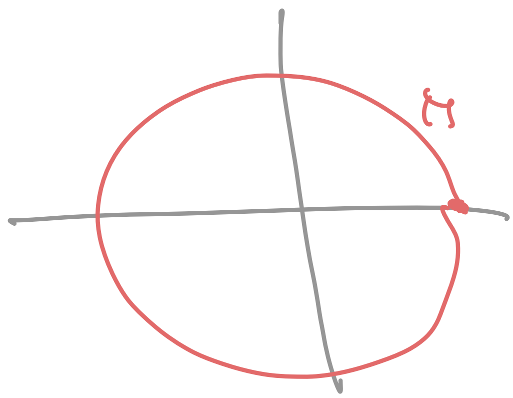
1. Sternförmige Menge ✓



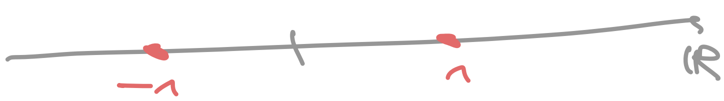
2. $S^n = \{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \}$

einfach zshg für $n \geq 2$.

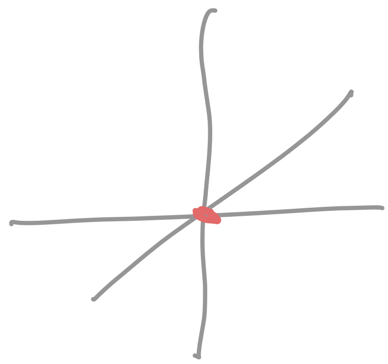
Für $n = 1$ nicht:



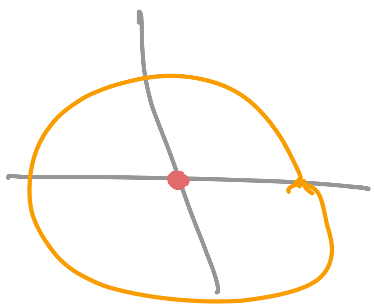
Für $n = 0$:



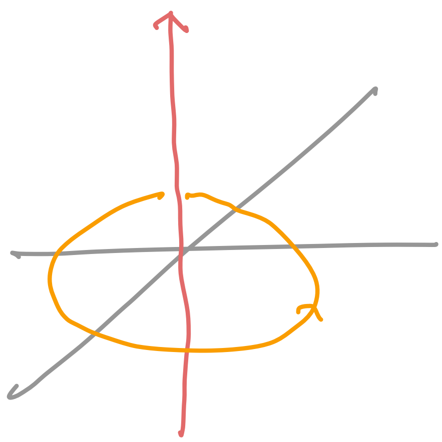
3. $\mathbb{R}^3 \setminus \{0\}$

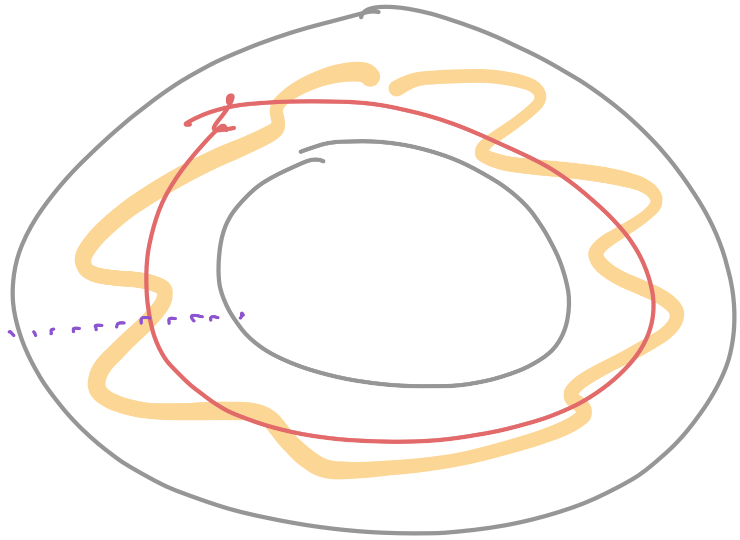


4. $\mathbb{R}^2 \setminus \{0\}$ <u>nicht</u> 1-zshg.

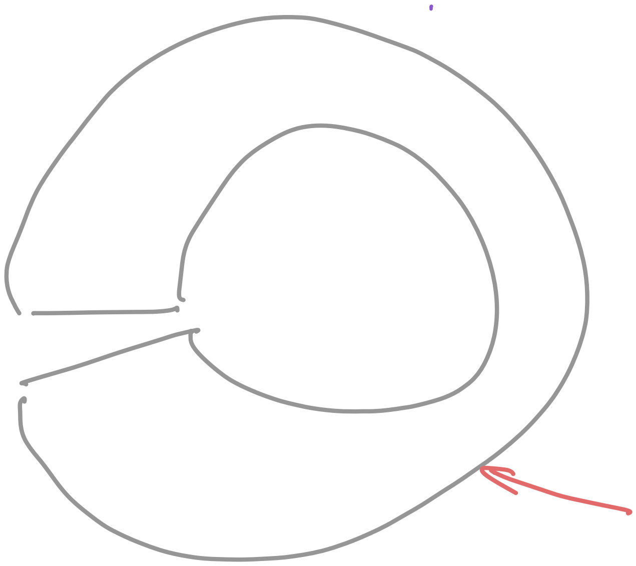


5. $\mathbb{R}^3 \setminus$ Gerade





nicht 1-fzhg



1 fzhg.

Beweis:

$$\nabla \times U = 0 \qquad \text{"Rotation } 0\text{"}$$

$\Rightarrow \quad \alpha_U = \langle U, \cdot \rangle$ geschlossen

$\Rightarrow \quad \alpha_U$ lokal exakt

Gebiet einfach zshg:

$\Rightarrow \quad \alpha_U$ global exakt:

$$\alpha_U = du$$

Übersetzung: $V = \nabla u$.