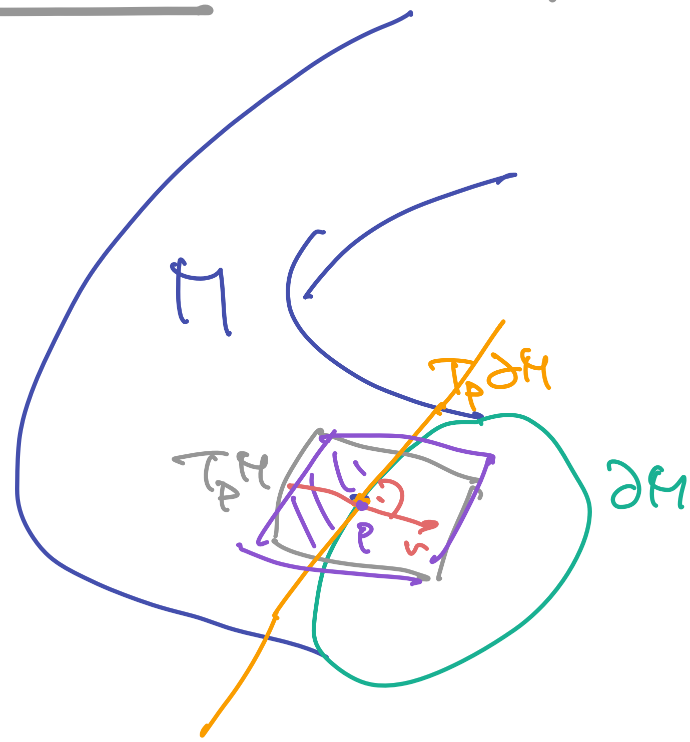
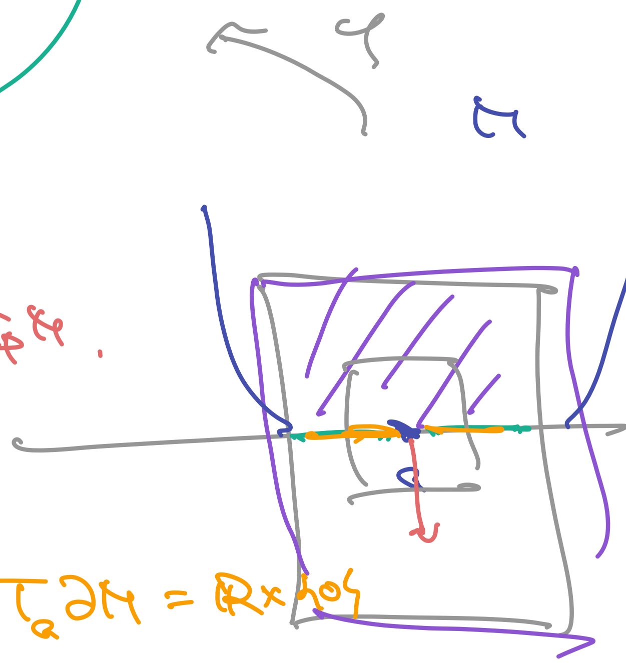
Vü-9





$$T_p\partial M \oplus \langle n \rangle = T_pM.$$

$$T_0M = \mathbb{R}\times\mathbb{R} \supset T_0\partial M = \mathbb{R}\times\{0\}$$

$$M = f^{-1}(0) \subset \mathbb{R}^{n+m}$$

$$f = (f_1, \dots, f_m)^T$$

$$\nabla f_j \perp TM \; .$$

__Definition:__

$$(v_1, \dots, v_n)$$

$$\sim [w_1, \dots, w_n]$$

$$\iff \det A > 0 \,, \quad \text{wobei}$$

$$A v_j = w_j \,, \quad 1 \le j \le n$$

Das ist wohldefiniert:

$$(w_1, \dots, w_n) = A \underbrace{[v_1, \dots, v_n]}_{\text{Matrix mit Spalten } v_1, \dots, v_n}$$

$$\underset{2}{\det(\quad)} = \underbrace{\det A}_{(!)} \cdot \underset{2}{\det(\quad)}$$

$\det A > 0$,

gleiche Vorzeichen

$f_j \equiv \text{const} \equiv 0$ auf M

$\Rightarrow \quad \nabla f_j(p) \perp T_pM$

Denn $\quad f_j(\varphi(t)) \equiv 0$

φ Kurve auf M

durch p

$$\Rightarrow \quad \frac{d}{dt}\Big|_{t=0} : \quad Df_j \cdot \dot\varphi(0) = 0$$

$$\langle \nabla f_j(p), v \rangle$$

Tangentialvektor

$\Rightarrow \quad \nabla f_j(p) \perp T_pM$

1 Man verifiziere den Satz von Stokes, also

$$\int_c \mathrm{d}\omega = \int_{\partial c} \omega,$$

für die 0-Form

$$\omega \in \Omega^0(\mathbb{R}^2), \quad \omega(x, y) = xy + x$$

und den 1-Würfel

$$c : \mathbb{I} \to \mathbb{R}^2, \quad c(t) = (\cos(\pi t), \sin(\pi t)).$$

Halbkreis

ω 0-Form $xy + x = x(y+1)$.

Dann:

$$\int_c d\omega = \int_{\mathbb{I}} c^*\big((y+1)\,dx + x\,dy\big)$$

$$= \int_0^1 (\sin(\pi t) + 1)\, d(\cos \pi t) + \cos \pi t \cdot d(\sin \pi t)$$

$-\pi \cdot \sin \pi t$ $\quad$ $-\pi$ $\quad$ $\cos \pi t$

$+ \quad = 0$

$$= \int_0^1 d(\cos \pi t)$$

$$= \cos \pi t \Big|_0^1 = -2.$$

$$\int_{\partial c} \omega \quad : \quad \partial c = c_{1,1} - c_{1,0}$$

$$\int_{\partial c} \omega = \int_{\underbrace{c_{1,1}}_{\text{Punkte}}} \overbrace{\omega}^{\text{0-Form}} - \int_{\underbrace{c_{1,0}}} \overbrace{\omega}$$

$$= \omega(c(1)) - \omega(c(0))$$

$$= \omega(-1,0) - \omega(1,0)$$

$\omega = x(y+1)$

$$= -1 - 1$$

$$= -2. \qquad \blacksquare$$

2 Man verifiziere den Satz von Stokes für die 1-Form

$$\omega \in \Omega^1(\mathbb{R}^2), \quad \omega(x,y) = y\,\mathrm{d}x$$

und den 2-Würfel

$$c:\ \mathbb{I}^2 \to \mathbb{R}^2, \quad c(x,y) = (\cos(\pi x), y\sin(\pi x)).$$

$$\omega = y\,dx$$

$$c(x,y) = (\cos(\pi x),\ y \cdot \sin(\pi x))$$

Dann:

$$\int_c d\omega = \int_c dy \wedge dx$$

$$= \int_{\mathbb{I}^2} c^*(dy \wedge dx)$$

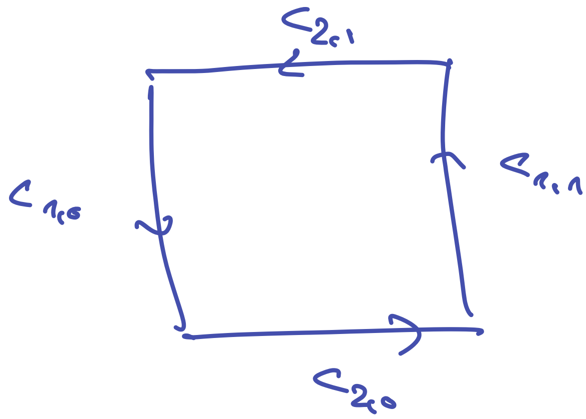
$$= \int_{\mathbb{I}^2} \underbrace{d(y \cdot \sin(\pi x))}_{dy \cdot \sin(\pi x)} \wedge \underbrace{d(\cos(\pi x))}_{\wedge(-\pi \sin \pi x\, dx)}$$

$$= -\pi \int_{\mathbb{I}^2} \sin^2 \pi x\ dy \wedge dx$$

$$= \pi \int_0^1 \int_0^1 \sin^2 \pi x\ d\lambda_2$$

$$= \pi \underbrace{\int_0^1 \sin^2 \pi x\, dx}_{\frac{1}{2}} = \frac{\pi}{2}.$$

$$\partial c = c_{1,1} - c_{1,0} + c_{2,0} - c_{2,1}$$



$y\,dx$ $\qquad c(x,y) = (\cos \pi x,\ y \sin \pi x)$

auf $c_{2,0}$: 0

$c_{2,1}$: $\sin \pi x \cdot d(\cos \pi x)$

$c_{1,0}$: 0

$c_{1,1}$: 0.

Also:

$$\int_{\partial c} \omega = -\int_{c_{2,1}} \omega$$

$$= -\int_0^1 \sin \pi x \, d(\cos \pi x)$$

$$= \pi \int_0^1 \sin^2 \pi x \, dx = \frac{\pi}{2}.$$

3 Sei

$$\omega = \mathrm{d}x_{i_1} \wedge .. \wedge \mathrm{d}x_{i_k} \in \Omega^k(\mathbb{R}^n).$$

Für eine differenzierbare Abbildung

$$\varphi = (\varphi_1, .., \varphi_n)^\top : \mathbb{R}^m \hookrightarrow \mathbb{R}^n$$

gilt

$$\varphi^*\omega = \sum_{1 \leqslant j_1 < .. < j_k \leqslant m} \left(\sum_{\sigma \in \mathcal{P}_k} \operatorname{sgn}(\sigma) \frac{\partial \varphi_{i_1}}{\partial x_{j_{\sigma(1)}}} .. \frac{\partial \varphi_{i_k}}{\partial x_{j_{\sigma(k)}}} \right) \mathrm{d}x_{j_1} \wedge .. \wedge \mathrm{d}x_{j_k}.$$

$$d\varphi_{i_1} = \underbrace{\sum_{j=1}^{m} \frac{\partial \varphi_{i_1}}{\partial x_j} dx_j}$$

Idee:

$$\varphi^*\omega = \varphi^*(dx_{i_1} \wedge \dots \wedge dx_{i_k})$$

$$= \varphi^* dx_{i_1} \wedge \dots \wedge \varphi^* dx_{i_k}$$

$$= \underbrace{d\varphi_{i_1}} \wedge \dots \wedge \underbrace{d\varphi_{i_k}}$$

$$= \sum_{\substack{1 \le j_1, \dots, j_k \le n \\ \text{ungeordnet}}} \frac{\partial \varphi_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial \varphi_{i_k}}{\partial x_{j_k}} \cdot \underbrace{dx_{j_1} \wedge \dots \wedge dx_{j_k}}$$

$$= \sum_{j_1 < j_2 < \dots < j_k} \left(\sum_{\sigma \in \mathcal{P}_k} \operatorname{sgn}(\sigma) \frac{\partial \varphi_{i_1}}{\partial x_{j_{\sigma(1)}}} \cdots \frac{\partial \varphi_{i_k}}{\partial x_{j_{\sigma(k)}}} \right) dx_{j_1} \wedge \dots \wedge dx_{j_k}.$$

4 Was erhält man in der vorangehenden Aufgabe für
a. $m = 1$, $k = 1$, $n = 3$ und $i_1 = 2$?
b. $m = 2$, $k = 1$, $n = 3$ und $i_1 = 2$?
c. $m = 2$, $k = 2$, $n = 3$ und $i_1 = 1$, $i_2 = 3$?

$n = 3$: Alles in $\mathbb{R}^3 \ni (x, y, z)$

a. $k = 1$, 1-Form, $i_1 = 2$:

$\omega = dy$

$m = 1$: $\varphi : \mathbb{R}^1 \to \mathbb{R}^3$, $\varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \vdots \\ \varphi_3(t) \end{pmatrix}$

Dann.

$$\varphi^* \omega = \varphi^*(dy)$$
$$= d\varphi_2$$
$$= \dot\varphi_2(t)\, dt$$

b. $\omega = dy$ wie in a.

$m = 2$: $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$,

$$\varphi(x, y) = \begin{pmatrix} \varphi_1(x, y) \\ \vdots \\ \vdots \end{pmatrix}$$

Dém.:

$$\varphi^* \omega = d\varphi_2$$

$$= \varphi_{2,x}\, dx + \varphi_{2,y}\, dy.$$

Rapp. voir Rappels 3.

Ex: $k = 2$: 2-forme sur $\mathbb{R}^3$

$i_1 = 1$, $i_2 = 3$:

$$\underset{i_1}{dx} \wedge \underset{i_3}{dz}$$

$n = 2$: φ une fct. de 2. :

Dém.:

$$\varphi^* \omega = \varphi^* (dx \wedge dz)$$

$$= d\varphi_1 \wedge d\varphi_3$$

$$= (\varphi_{1,x}\, dx + \varphi_{1,y}\, dy) \wedge (\varphi_{3,x}\, dx + \varphi_{3,y}\, dy)$$

$$= (\varphi_{1,x}\varphi_{3,y} - \varphi_{1,y}\varphi_{3,x})\, dx \wedge dy$$