VÜ-3 27.11.2020

1. e) $\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$

$(a-b)(a+b)$
$= a^2 - b^2$

$\sqrt{n+1}+\sqrt{n}$

$$= \sqrt{n}\,\frac{(\quad)(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$$

$$= \sqrt{n}\,\frac{(n+1)-(n)}{\sqrt{n+1}+\sqrt{n}}$$

$$= \sqrt{n}\,\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$= \cancel{\frac{\sqrt{n}}{\sqrt{n}}}\cdot\frac{1}{\underbrace{\sqrt{1+\frac{1}{n}}}_{\to 1}+1} \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

b. $\frac{n}{\sqrt{n+1}} - \sqrt{n}$

$$= \frac{n - \sqrt{n}\cdot\sqrt{n+1}}{\sqrt{n+1}}$$

$$= \frac{n - \sqrt{n^2+n}}{\sqrt{n+1}}$$

$$= n\,\frac{1-\sqrt{1+\frac{1}{n}}}{\sqrt{n+1}}$$

$$= \frac{n}{\sqrt{n+1}} \cdot \frac{\overbrace{1-\left(1+\frac{1}{n}\right)}^{-\frac{1}{n}}}{1+\sqrt{1+\frac{1}{n}}}$$

$$= -\underbrace{\frac{1}{\sqrt{n+1}}}_{\to 0} \cdot \frac{1}{1+\underbrace{\sqrt{1+\frac{1}{n}}}_{\to 1}} \xrightarrow[n\to\infty]{} 0$$

c. $\sqrt{n+\sqrt{n}} - \sqrt{n-\sqrt{n}}$

$$= \frac{(n+\sqrt{n}) - (n-\sqrt{n})}{\sqrt{n+\sqrt{n}} + \sqrt{n-\sqrt{n}}}$$

$$= \frac{2\sqrt{n}}{\sqrt{n+\sqrt{n}} + \sqrt{n-\sqrt{n}}}$$

$$= \frac{2\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(\underbrace{\sqrt{1+\frac{1}{\sqrt{n}}}}_{\to 1} + \underbrace{\sqrt{1-\frac{1}{\sqrt{n}}}}_{\to 1}\right)}$$

$$\xrightarrow[n\to\infty]{} \frac{2}{\sqrt{1}+\sqrt{1}} = 1. \quad \blacksquare$$

2. Vor: $a_n \longrightarrow a$

$$s_n = \frac{1}{n} \sum_{k=1}^{n} a_k$$

Z.z.: $s_n \longrightarrow a$.

Sei $\varepsilon > 0$.

$$|s_n - a| = \left| \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right) - a \right|$$

$$= \frac{1}{n} \left| \underbrace{\left(\sum_{k=1}^{n} a_k \right)}_{n \text{ Summanden}} - na \right|$$

$$= \frac{1}{n} \left| \sum_{k=1}^{n} (a_k - a) \right|$$

$$\leq \underbrace{\frac{1}{n}}_{\to 0} \sum_{k=1}^{n} |a_k - a| .$$

$|a_k - a| < \varepsilon$ für $k \geq N$

Zu $\frac{\varepsilon}{2}$ ex. N:

$$|a_k - a| < \frac{\varepsilon}{2}, \quad k \geq N$$

Betrachte Summe:

$$\frac{1}{n} \underbrace{\sum_{k=1}^{N} |a_k - a|}_{\text{fest}} < \frac{\varepsilon}{2}, \quad n \geq N_*$$

Also uniforment :

$$|s_n - a| \le \frac{1}{n} \sum_{k=1}^{n} |a_k - a|$$

für $n > N_\varepsilon$: und $n \ge N$

$$\le \frac{1}{n} \left(\sum_{k=1}^{N} |a_k - a| + \sum_{k=N+1}^{n} \underbrace{|a_k - a|}_{\frac{\varepsilon}{2}} \right)$$

$$\le \underbrace{\frac{1}{n} \sum_{k=1}^{N} |a_k - a|} + \frac{1}{n} \sum_{k=N+1}^{n} \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon .$$

$$a_n = (-1)^n$$

$$s_n = \sum_{k=1}^{n} (-1)^k \longrightarrow 0 .$$

3. Esempi :

1. a_n sia monotona crescente: $a_n \nearrow a$

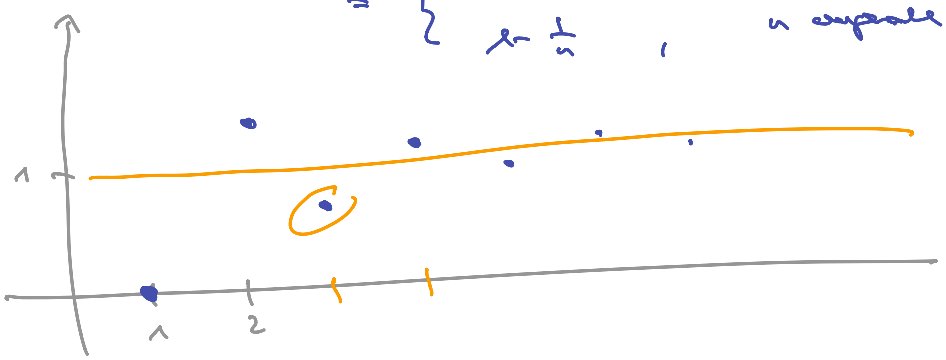
Allora

$$b_n = \inf\{a_k : k \geq n\} = a_n$$

$$c_n = \sup\{ \text{———} \} = a.$$

2. $$a_n = 1 + \frac{(-1)^n}{n} \longrightarrow 1$$

$$= \begin{cases} 1 + \frac{1}{n}, & n \text{ pari} \\ 1 - \frac{1}{n}, & n \text{ dispari} \end{cases}$$



Allora $$b_n = \sup\{a_k : k \geq n\}$$

$$= \begin{cases} 1 - \frac{1}{n}, & n \text{ dispari} \\ 1 - \frac{1}{n+1}, & n \text{ pari}. \end{cases}$$

3. $a_n = (-1)^n$:

$$b_n = -1 \longrightarrow -1$$

$$c_n = 1 \longrightarrow 1 .$$

<u>Zeige</u>: Betrachte c_n.

Wissen: $a_n \longrightarrow a$,

D.h.

$$|a_k - a| < \varepsilon, \quad k \geq N(\varepsilon)$$

$$\Longleftrightarrow \quad a-\varepsilon < a_k < a+\varepsilon, \quad k \geq N(\varepsilon)$$

Sei $n \geq N(\varepsilon)$:

$$a-\varepsilon < a_k < a+\varepsilon, \quad k \geq n \geq N(\varepsilon)$$

$$a-\varepsilon \leq \underbrace{\sup\{a_k : k \geq n\}}_{c_n} \leq a+\varepsilon$$

Also:

$$|c_n - a| \leq \varepsilon, \quad n \geq N(\varepsilon).$$

Also gilt: $c_n \longrightarrow a, \quad n \to \infty$

Achtung: Sei (a_n) beschränkt.

Dann ex.

$$b = \inf \{ a_n : n \geq 1 \}$$

$$c = \sup \{ a_n : n \geq 1 \}.$$

Dann existiert nicht notwendig eine Teilfolge von (a_n), die gegen b oder c konvergiert.

Sondern: Limes inferior

Limes superior.

4\. a) False.

$a_n = 1, \quad b_n = (-1)^n$

$a_n b_n = (-1)^n$

b) ✓

c) False:

$a_n = b_n = (-1)^n$

$a_n b_n = (-1)^{2n} = 1$

d) ✓

5. $\lim \left(1+\frac{1}{n}\right)^n$ [illegible] [illegible]

$= \left(\lim \left(1+\frac{1}{n}\right)\right)^n$ — [illegible],

$= 1^n$

$= 1.$

[illegible]: $\left(1+\frac{1}{n}\right)^n$ [illegible] [illegible] [illegible]:

$1 < \frac{9}{4} < \ldots < 3$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ [illegible].