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Funktionalanalysis (WS 2018/19)

Gruppenübungsblatt 1 (Bonus)

Exercise 1.1.

Let S be a nonempty set and endow the vector space

$$B(S,\mathbb{R}) := \{f\colon S \to \mathbb{R} \mid f \text{ is bounded}\},$$

with the *uniform metric* $d_\infty\colon B(S,\mathbb{R}) \times B(S,\mathbb{R}) \to \mathbb{R}$ given by

$$d_\infty(f,g) := \sup_{x\in S} |f(x) - g(x)|.$$

(a) Show that d_∞ defines indeed a metric on the vector space $B(S,\mathbb{R})$.

(b) Prove that $B(S,\mathbb{R})$ is complete with respect to d_∞.

(c) Suppose (S,d) is a metric space and consider the linear subspace $C_b^0(S,\mathbb{R})$ of $B(S,\mathbb{R})$, consisting of continuous and bounded functions $f\colon S \to \mathbb{R}$. Prove that $C_b^0(S,\mathbb{R})$ is closed in $B(S,\mathbb{R})$.

(d) Is $C_b^0(S,\mathbb{R})$ complete with respect to d_∞? Motivate your answer.

Exercise 1.2.

Let (M,d) be a metric space. Let $x \in M$ and $r > 0$. It is not true in general that the closure of the open ball $B_x(r)$ equals the set $\{y \in M \mid d(x,y) \le r\}$. Show this by giving an example.

Exercise 1.3.

Let $a,b \in \mathbb{R}$ with $a < b$. Let $\{f_n\}$ be a sequence in $C([a,b],\mathbb{R})$ which is pointwise bounded, that is, for every $x \in [a,b]$ there exists $M_x > 0$ such that $|f_n(x)| \le M_x$ for all $n \in \mathbb{N}$. Prove that there exists a subinterval of $[a,b]$ on which the f_n are uniformly bounded, that is, there exist $c,d \in [a,b]$ with $c < d$ and $M > 0$ such that $|f_n(x)| \le M$ for all $x \in [a,b]$ and $n \in \mathbb{N}$.

Hint: apply Baire's Theorem.

Exercise 1.4.

Which of the following formulas define a metric? Give for each case a proof or a counterexample.

(a) $d(f,g) = \sup_{x\in[-1,1]} |x||f(x) - g(x)|$ on $B([-1,1],\mathbb{R})$ (see exercise 1.1.).

(b) $d(f,g) = \int_{-1}^{1} |x|\,|f(x) - g(x)|\mathrm{d}x$ on $C^0([-1,1],\mathbb{R})$.

(c) $d(f,g) = \int_{-1}^{1} |x|\,|f'(x) - g'(x)|\mathrm{d}x$ on $C^1([-1,1],\mathbb{R})$.

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Exercise 1.5.

Consider the scalar Riccati equation

$$\begin{aligned} \frac{du}{dt} &= \epsilon u^2 + a(t)u + b(t), \\ u(0) &= u_0. \end{aligned} \tag{1}$$

where $a, b \colon \mathbb{R} \to \mathbb{R}$ are continuous functions and $u_0, \epsilon > 0$ are real numbers.

(a) Prove that (1) has a unique local solution, i.e. there exists $t_0 > 0$ such that (1) has a unique solution $u \in C^1([0, t_0), \mathbb{R})$.

Hint: apply the Picard-Lindelöf theorem.

(b) Show that there exists $a, b \colon \mathbb{R} \to \mathbb{R}$ such that for each $\epsilon > 0$ the local solution to (1) is non-global, i.e. the solution does not lie in $C^1([0, \infty), \mathbb{R})$.

(c) Prove that if $u \in C^0([0, \infty), \mathbb{R})$ solves the integral equation

$$u(t) = \int_0^t e^{\int_s^t a(r)\mathrm{d}r} \left(\epsilon u(s)^2 + b(s)\right) \mathrm{d}s + e^{\int_0^t a(r)\mathrm{d}r} u_0, \qquad t \in [0, \infty), \tag{2}$$

then u is a (global) solution in $C^1([0, \infty), \mathbb{R})$ to (1).

(d) Suppose $b \colon \mathbb{R} \to \mathbb{R}$ is bounded and $a \colon \mathbb{R} \to \mathbb{R}$ is bounded from above by a negative constant, i.e. there exist constants $C > 0$ and $a_0 < 0$ such that $|b(t)| \leq C$ and $a(t) \leq a_0$ for all $t \in \mathbb{R}$. Prove that there exists $\epsilon_0 > 0$ such that (2) has a global solution in $C_b^0([0, \infty), \mathbb{R})$ for each $\epsilon \in (0, \epsilon_0)$.

Hint: show that the operator $F \colon C_b^0([0, \infty), \mathbb{R}) \to C_b^0([0, \infty), \mathbb{R})$, given by

$$F(u)(t) = \int_0^t e^{\int_s^t a(r)\mathrm{d}r} \left(\epsilon u(s)^2 + b(s)\right) \mathrm{d}s + e^{\int_0^t a(r)\mathrm{d}r} u_0, \qquad t \in [0, \infty),$$

restricts to a contraction on a closed ball $\overline{B_0(\rho)} \subset C_b^0([0, \infty), \mathbb{R})$ for some $\rho > 0$.

Notation: Let $k > 0$ and $m \geq 0$ be integers and let $\Omega \subset \mathbb{R}^k$. We denote by $C^m(\Omega, \mathbb{R})$ the vector space of continuous functions $f \colon \Omega \to \mathbb{R}$, which are m times continuously differentiable. Moreover, $C_b^0(\Omega, \mathbb{R})$ is the vector space of bounded and continuous functions $f \colon \Omega \to \mathbb{R}$.

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