Eigenvectors of Linear Transformations

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The following examples attempt to show linear transformations at work, represented as multiplication by matrices A, B, C and D, respectively. We will indicate a vector v (in black) together with its image (in yellow). Moreover, a triangle (pale green) is shown together with its image under the transformation. Note that we follow Poole (Linear algebra, Boston, 2011) in calling every linear map a 'transformation', irrespective of its invertibility. In the examples below, we will give the effect of our geometric transformation by applying matrices to columns. These columns describe the position vectors with respect to a coordinate system that will be indicated if necessary.

The one-dimensional space span(v) is indicated by a *very* thick gray line in the drawings.

In the interactive HTML-version[1] you may use your mouse to move v (drag its tip); the image of v under the respective transformation will move automatically.

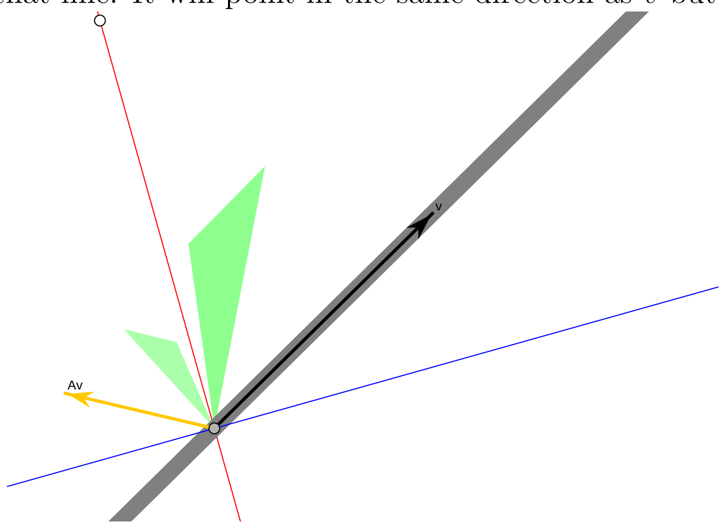
1 Reflection, followed by scaling

Our black vector v is mapped to the yellow vector Av by reflecting in the thin red line and then scaling by the factor 1/2.

If we introduce cartesian coordinates with axes as indicated by the red and blue line, this transformation is affected by multiplication with the matrix (product)

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

Turn the black vector v such that it lies on the red line. Then the image Av will also lie on that line. It will point in the same direction as v but has only half of its length:



We have thus found an eigenvector for the eigenvalue $\frac{1}{2}$.
This relation is preserved if you scale the vector v but keep it on the red line.

If you move v such that it lies on the blue line then the image Av will also lie on that line. Again, the image will have half of the length of v but it will point in the opposite direction:

We have thus found an eigenvector for the eigenvalue $-\frac{1}{2}$.

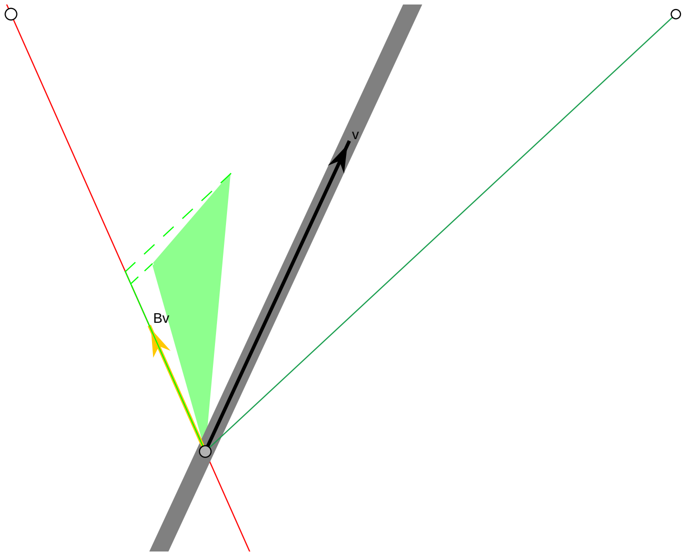
[1] You find that version using the address
`https://info.mathematik.uni-stuttgart.de/HM-Stroppel-Material/Eigenvectors/`.

It is just a coincidence (in fact, a consequence of the fact that A is *symmetric*) that the eigenspaces are orthogonal to each other).

The interactive version also allows to move the right line.

2 Projection (that is, a non-invertible transformation)

In the following example we map v to Bv obtained by projection along the direction of the green line, onto the red line[2]. Find the directions of the eigenvectors!



Vectors pointing along the red line will not be changed at all by this projection.

These are eigenvectors for the eigenvalue 1.

Vectors pointing along the green line (viz, the direction of our projection) are eigenvector for the eigenvalue 0:

These vectors are mapped to the zero vector.

Matrix examples for such projections include the following.

- $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

  (if the green and red lines form the first and second axis of a cartesian coordinate system, respectively).

- $B = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix}$

  (if the red line is spanned by the vector $u = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$, and the green line is perpendicular to the red one).

- $B = \begin{bmatrix} 5 & -2 \\ 10 & -4 \end{bmatrix}$

  (here the red line is spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the green one by $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$; these lines are *not* perpendicular).

[2] You may change these directions in the interactive version.

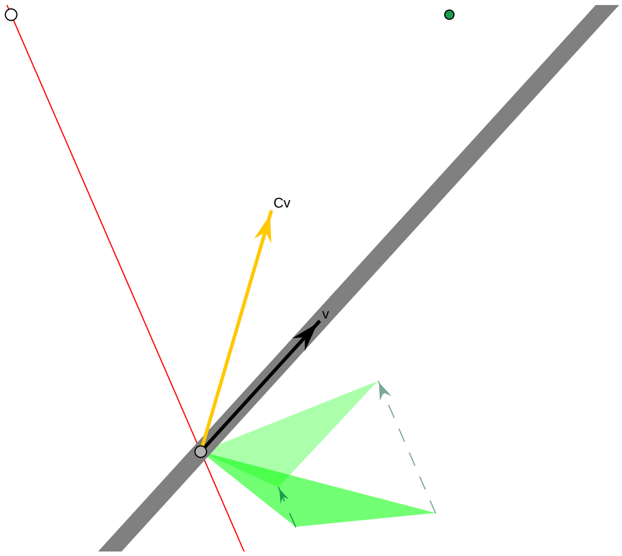
There are many more examples of 2×2 matrices with eigenvalues 0 and 1; in fact, these are just those matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a + d = 1$ and $ad - bc = 0$.

In other words, the trace and determinant of B have to satisfy $\operatorname{tr}(B) = 1$ and $\det(B) = 0$. This may also be phrased as: *The characterstic polynomial* $\det(B - \lambda I)$ *of* B *equals* $\lambda^2 - \lambda$.

3 Shearing (algebraic and geometric multiplicities differ)

In the following example, we do not find (in fact, we do not have) eigenvectors in different directions.

Our shearing fixes each point on the red line[3]; every other point is then moved on the parallel to the red line through that point. Find the directions of the eigenvectors!



Vectors pointing along the red line will not be changed by the shear:

These are eigenvectors for the eigenvalue 1.

As every other vector changes his direction under the shear, there are *no other* eigenvectors.

Among the matrices describing such shears, we have:

- $C = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ (here the red line is the second coordinate axis).
- $C = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$ (the red line is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.)

The characteristic property of these matrices describing shears is the fact that they have the eigenvalue 1 with algebraic multiplicity 2 while the geometric multiplicity is only 1.

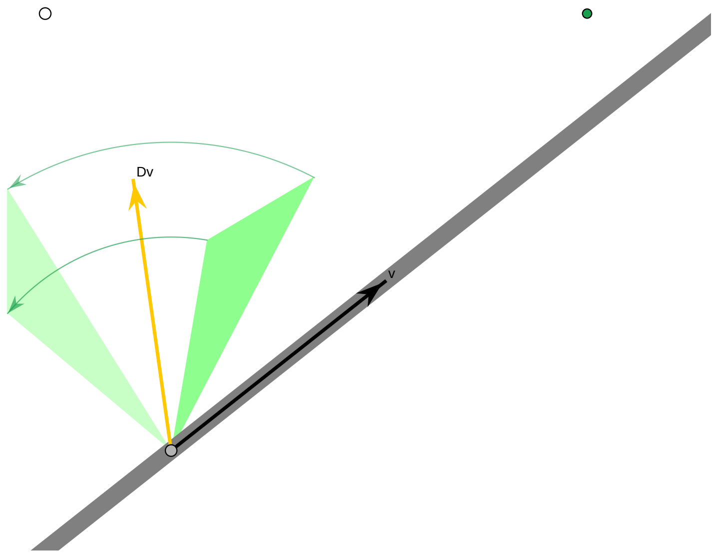
There are many more examples of 2×2 matrices with this property; in fact, these are just those matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that are not diagonal (i.e., at least one of the entries b, c is different from zero) with $a + d = 2$ and $ad - bc = 1$.

In other words, the trace and determinant of C have to satisfy $\operatorname{tr}(C) = 2$ and $\det(C) = 1$. This may also be phrased as follows:
The characterstic polynomial $\det(C - \lambda I)$ *of* C *equals* $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$.

[3] You may change this line and the amount of shearing in the interactive version.

4 Rotation (no real eigenvalues, but imaginary ones)

Finally, we discuss an example of a transformation that can be described by a matrix with real entries but having no real eigenvalue at all:



We rotate v.

(The interactive version allows to change the angle of rotation.)

Under the rotation indicated in the drawing, every non-zero vector will change its direction. Therefore, there are no eigenvectors to be seen.

However — can you think of other angles of rotation where we do see eigenvectors?

The drawing indicates a counterclockwise rotation by 60°, or $\frac{\pi}{3}$ radians. That rotation can be described by the matrix

$$D = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

The characteristic polynomial is $\det(D - \lambda I) = \lambda^2 - \lambda + 1$; the eigenvalues are $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

There are many more examples of 2×2 matrices without real eigenvalues; in fact, these are just those matrices $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where the characteristic equation

$$0 = \det(D - \lambda I) = \lambda^2 - (a + d)\lambda + (ad - bc)$$

has no solution in the field of real numbers.

This happens precisely if the discriminant $(a + d)^2 - 4(ad - bc)$ is negative; i.e., if

$$(a - d)^2 < -4bc\,.$$

(For the rotation in the drawing, we have $a = d$ and $bc < 0$, and the condition is satisfied.)