Hyperbolas defined by quadratic equations

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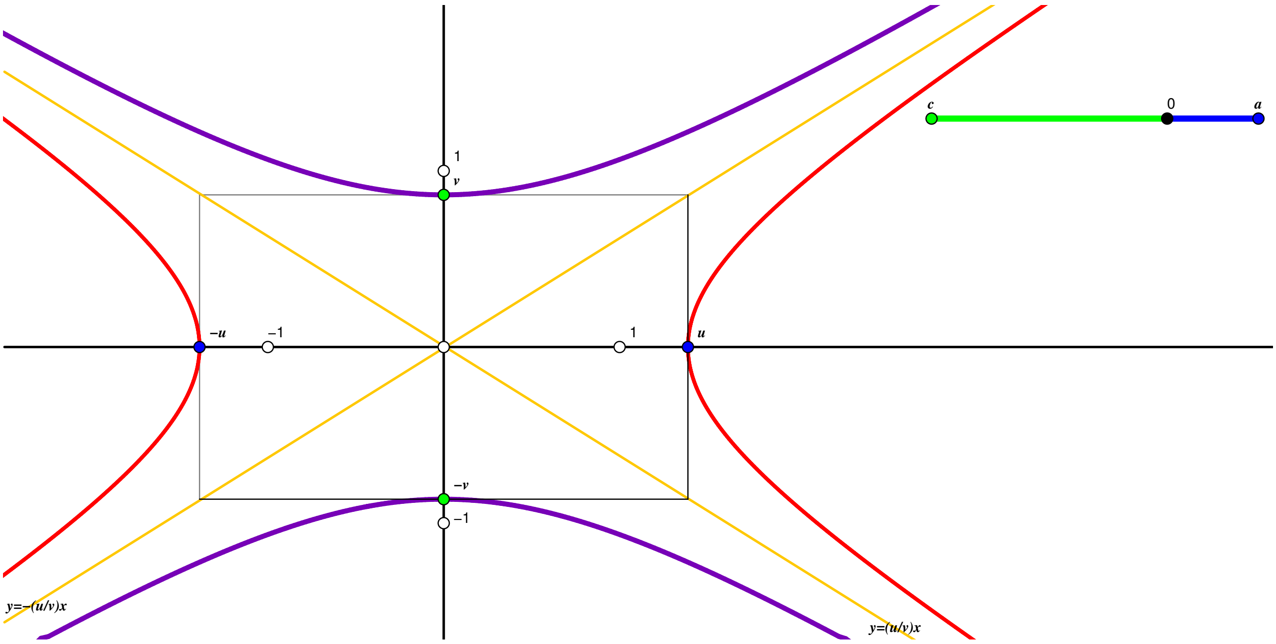
Playing with the equations

The following picture (also available as an interactive applet[1]) shows the two hyperbolas that are defined by the equations $ax^2 - cy^2 = 1$ (that is the red one) and $-ax^2 + cy^2 = 1$ (the dark violet one), respectively.

Also shown in the picture are the points with coordinates $u = 1/\sqrt{a}$ and $v = 1/\sqrt{c}$ on the coordinate axes (so our equations become $(\frac{x}{u})^2 - (\frac{y}{v})^2 = 1$ and $-(\frac{x}{u})^2 + (\frac{y}{v})^2 = 1$, respectively), and the pair of lines solving the equation $ax^2 - cy^2 = 0$ (that is, $(\frac{1}{u}x + \frac{1}{v}y)(\frac{1}{u}x - \frac{1}{v}y) = 0$); these lines are the asymptotes for both of the hyperbolas.



In the interactive applet, you may adjust the values of a and c by moving the respective points in the applet with your mouse - just be sure to keep a to the right and c to the left of the black point marked 0. The hyperbolas and their asymptotes will be changed accordingly.

This picture, and the applet itself, were produced with Cinderella.

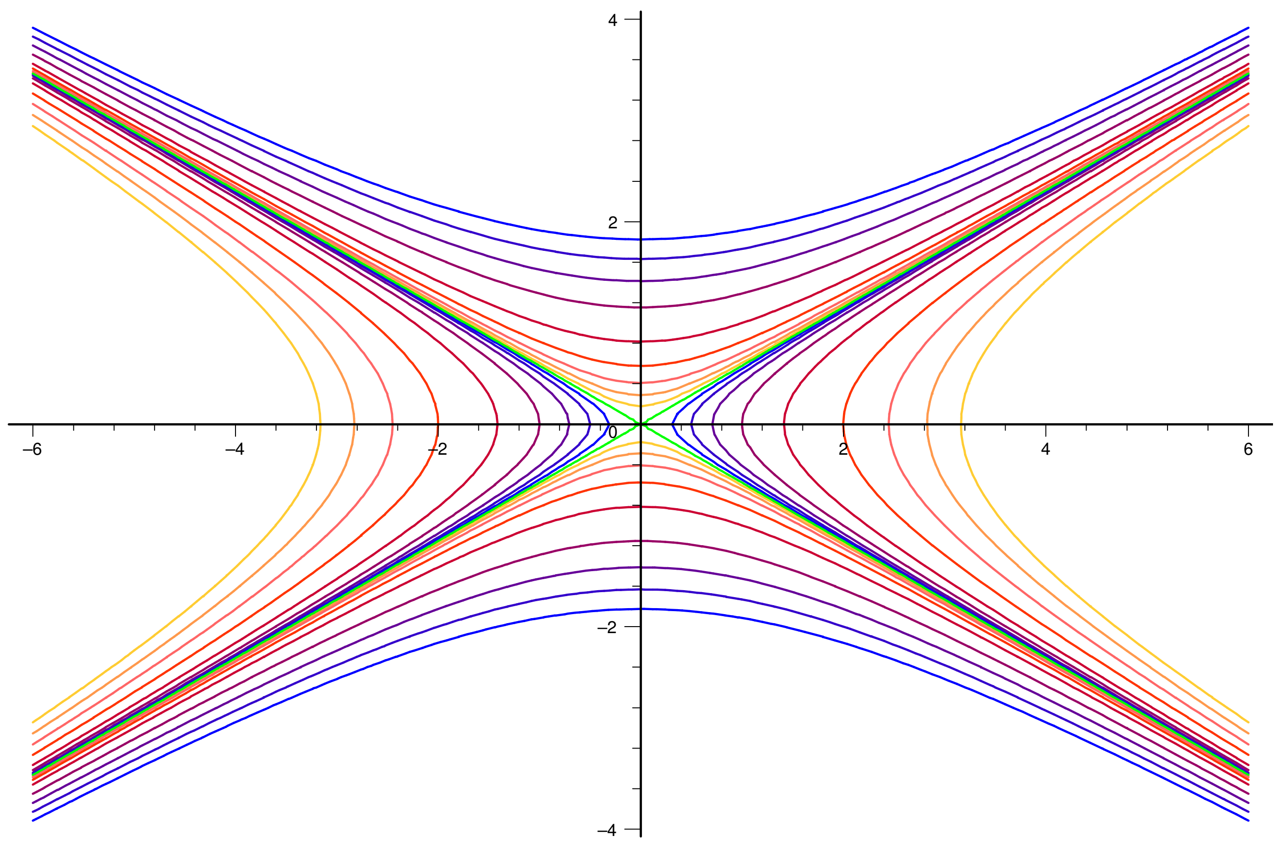
[1] You find that interactive version using the address
`https://info.mathematik.uni-stuttgart.de/HM-Stroppel-Material/Hyperbolas/`

A family of hyperbolas

The following picture shows many hyperbolas in one cartesian coordinate system; these hyperbolas may be described as curves consisting of the points solving an equation of the form $q(x, y) = d$ for different values of d (namely, $d \in \{0.1, 0.25, 0.5, 1, 2, 4, 6, 8, 10\}$ giving those hyperbolas that meet the horizontal axis, and $d \in \{-0.1, -0.25, -0.5, -1, -2, -4, -6, -8, -10\}$ giving those that meet the vertical axis) while $q(x, y) = x^2 - 3y^2$ in each case.

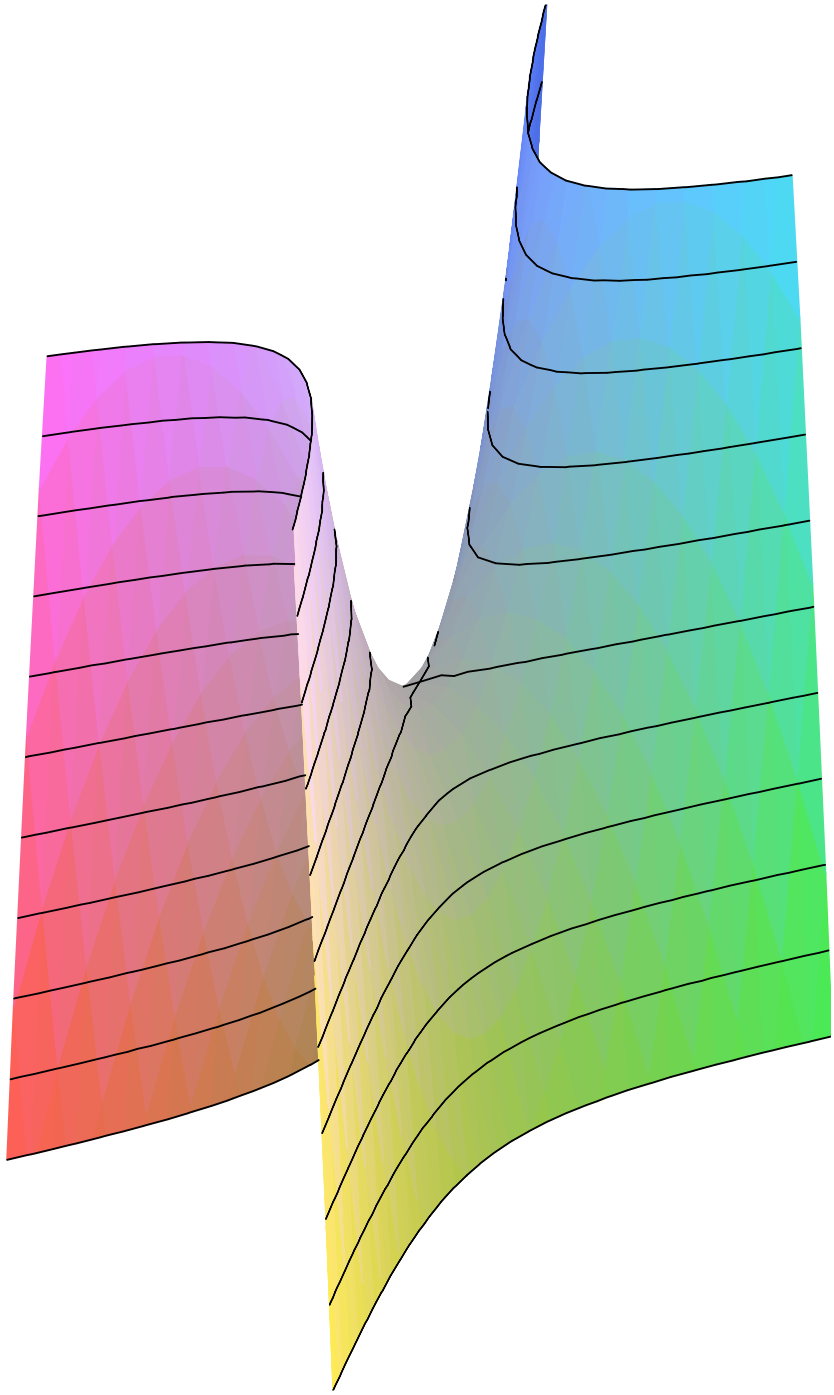
Instead of changing the value d, you may also keep d fixed and change the form q (just divide the whole equation by d).

The value $d = 0$ is special; it does not give the equation for a hyperbola but it gives the equation $q(x, y) = 0$ for a pair of lines (colored green in our picture). Actually, these lines form the so-called *asymptotes* for *each* hyperbola with equation $q(x, y) = d$ (i.e., with the same quadratic form q, and arbitrary constant $d \neq 0$). These asymptotes very nicely gives us the behaviour of the hyperbolas far away from the origin: the hyperbolas will approximate these lines.



We remark that each one of these hyperbolas may also be interpreted as a level set of the function

$q: \mathbb{R}^2 \to \mathbb{R}: (x, y) \mapsto x^2 - 3y^2$.



The picture on the right gives an impression of that graph; the black lines indicate the level sets

$\{(x, y) \in \mathbb{R}^2 \mid q(x, y) = 2n\}$

for each integer n within the range $-5 \leq d \leq 5$.

The surface obtained here as the graph of q is known as a *hyperbolic paraboloid*.

These pictures were produced using MAPLE®.