A quadric depending on a parameter

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The pictures

The pictures below (also available as an interactive applet[1]) show the quadric with equation

$$2\,x_1^2 + a\,x_1x_2 + x_2^2 = 1$$

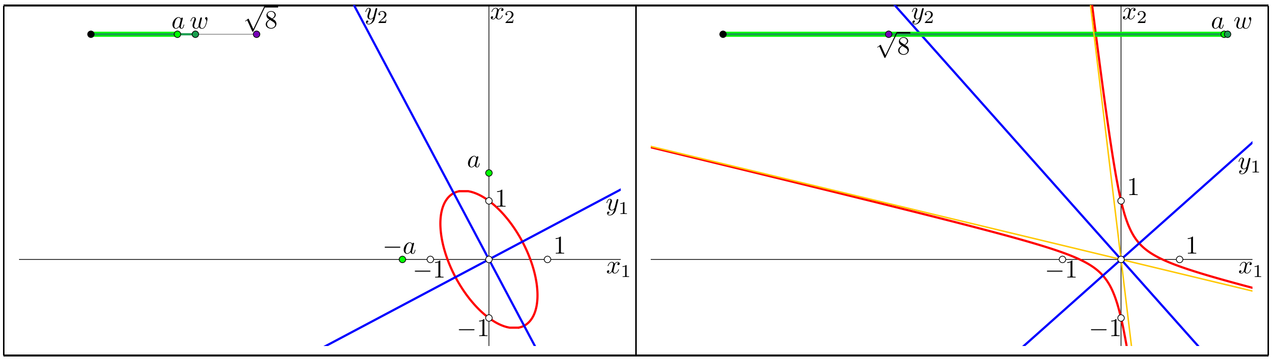
(in red), with the principal axes (in blue) for two special values of the parameter a.

In the interactive version, you may adjust the parameter a by dragging it with your mouse.

The quadric will be an ellipse as long as $|a| < \sqrt{8}$.

If $|a|$ reaches $\sqrt{8}$ (that's the point marked in dark violet) then the quadric degenerates to a pair of lines.

If $|a|$ exceeds $\sqrt{8}$ then the quadric becomes a hyperbola; the asymptotes will be shown in yellow.



The value $w = \sqrt{1+a^2}$ shown in the pictures will be used for the calculations in the next section.

[1] You find that interactive version using the address `https://info.mathematik.uni-stuttgart.de/HM-Stroppel-Material/Hyperbolas/quadric-depending-on-a.shtml`

Calculations

Using the symmetric matrix $A = \begin{bmatrix} 2 & a/2 \\ a/2 & 1 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we can write the equation as

$$\boldsymbol{x}^T A\, \boldsymbol{x} = 1\,.$$

The matrix A has characteristic polynomial $\lambda^2 - 3\lambda + 2 - a^2/4$; this gives the eigenvalues $\lambda_1 = (3 + \sqrt{1+a^2})/2$ and $\lambda_2 = (3 - \sqrt{1+a^2})/2$. We abbreviate $w = \sqrt{1+a^2}$ (this value is also shown on the slider that you use to adjust a in the picture).

If $a = 0$ then A is a diagonal matrix, and the equation is easy to understand (actually, our applet does not work well for $a = 0$ because a division by a is involved in the geometric construction behind the screen).

The eigenspaces are $E_{\lambda_1} = \operatorname{span}\left(\begin{bmatrix} a \\ w-1 \end{bmatrix}\right)$ and $E_{\lambda_2} = \operatorname{span}\left(\begin{bmatrix} 1-w \\ a \end{bmatrix}\right)$ (orthogonal to E_{λ_1}).

Using $N = \sqrt{a^2 + (w-1)^2} = \sqrt{2w(w-1)}$ we find the orthonormal basis

$$\mathcal{B} = \left\{ \frac{1}{N}\begin{bmatrix} a \\ w-1 \end{bmatrix}, \frac{1}{N}\begin{bmatrix} 1-w \\ a \end{bmatrix} \right\}$$

and the orthogonal matrix

$$Q = \frac{1}{N}\begin{bmatrix} a & 1-w \\ w-1 & a \end{bmatrix}$$

which in fact describes a rotation.

In coordinates $\boldsymbol{y} = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$ with respect to our new basis (i.e., for $\boldsymbol{x} = Q\boldsymbol{y}$), the equation of the quadric becomes easy: we have $\lambda_1\, y_1^2 + \lambda_2\, y_2^2 = 1$ because $\boldsymbol{x}^T A\, \boldsymbol{x} = \boldsymbol{y}^T (Q^T A\, Q)\, \boldsymbol{y}$ and $Q^T A\, Q = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$ is a diagonal matrix.

Whether this quadric is an ellipse or a hyperbola is decided by the signs of the eigenvalues: while λ_1 is clearly positive for any choice of a, the second eigenvalue λ_2 may be negative. This happens if, and only if, we have $|a| > \sqrt{8}$.

The boundary case $a = \sqrt{8}$ leads to the equation $3\, y_1^2 = 1$ which describes two lines parallel to the y_2-axis, at distance $1/\sqrt{3}$ from that axis.

You can *see* it all in the interactive version, go ahead and play with the value of a.

(Just in case you wondered whether $N = 0$ is possible: this only occurs if $a = 0$. In that case, the matrix A is diagonal, anyway; the eigenspaces are the standard axes. In our computation of the eigenspaces above, we actually would perform division by 0 if $a = 0$.)

The picture, and the applet itself, were produced with Cinderella and CindyJS.