Vortragsübung 3

Aufgabe V6:

a) $w = \cos\left(\frac{5}{3}\pi\right) + i\sin\left(\frac{5}{3}\pi\right)$

$z = \sqrt{3} - i$

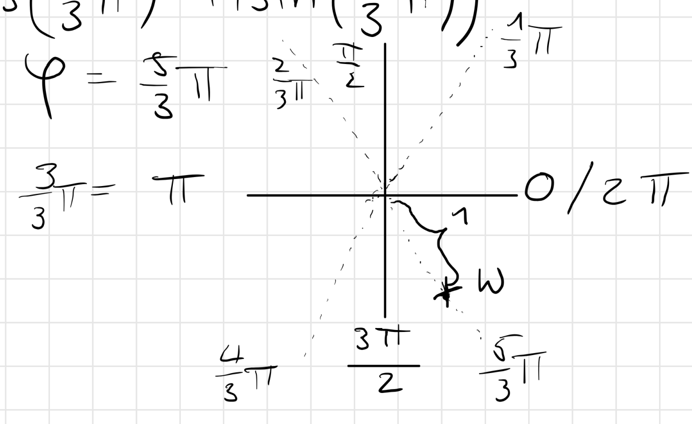
Erinnerung:
Für $z = a + bi \in \mathbb{C}$ gilt
$|z| := \sqrt{a^2 + b^2}$

$$|w| = \sqrt{\left(\sin\left(\frac{5}{3}\pi\right)\right)^2 + \left(\cos\left(\frac{5}{3}\pi\right)\right)^2}$$

$$= \sqrt{1} = 1$$

$$w = 1 \cdot \left(\cos\left(\frac{5}{3}\pi\right) + i\sin\left(\frac{5}{3}\pi\right)\right)$$

mit Winkel $\varphi = \frac{5}{3}\pi$



$$z = \sqrt{3} - i$$

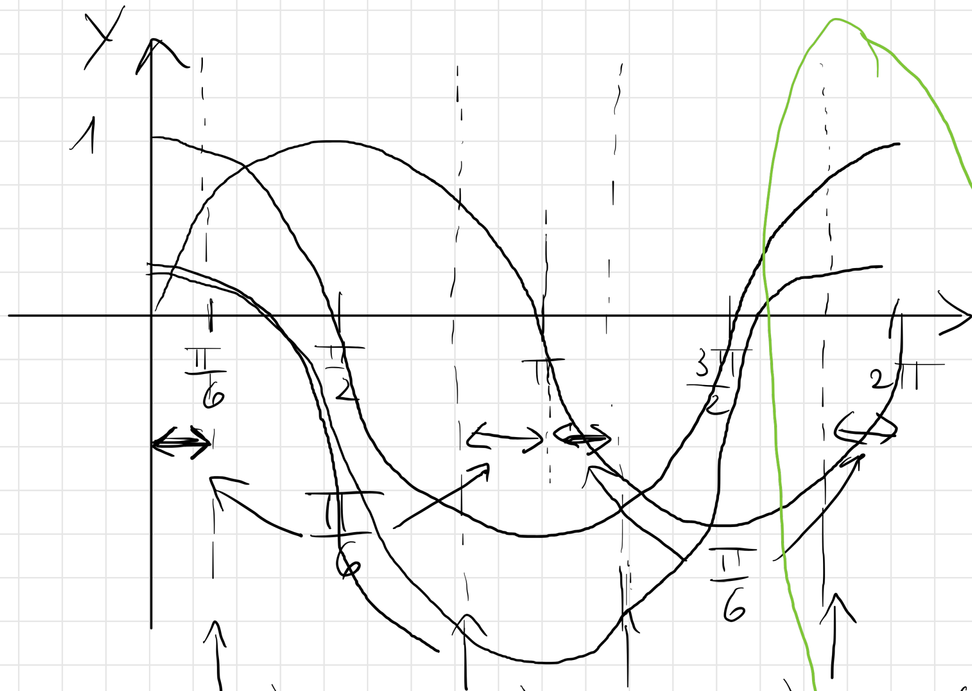
$$|z| = \sqrt{3+1} = 2$$

$$z = 2 \cdot \left(\frac{\sqrt{3}}{2} - \frac{1}{2} i \right)$$

$$\overset{*}{=} 2 \cdot \left(\cos\left(\frac{11}{6}\pi\right) + i \sin\left(\frac{11}{6}\pi\right) \right)$$

* Aus Tabelle:

$\frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$ und $\frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$



$$\sin\left(\frac{\pi}{6}\right) = \sin\left(\pi - \frac{\pi}{6}\right) = -\sin\left(\pi + \frac{\pi}{6}\right) = -\sin\left(2\pi - \frac{\pi}{6}\right)$$

$$\cos\left(\frac{\pi}{6}\right) = -\cos\left(\pi - \frac{\pi}{6}\right) = -\cos\left(\pi + \frac{\pi}{6}\right) = \cos\left(2\pi - \frac{\pi}{6}\right)$$

$$w = \cos\left(\frac{5}{3}\pi\right) + i\sin\left(\frac{5}{3}\pi\right)$$

$$w^5 = 1^5 \cdot \left(\cos\left(\frac{5}{3}\pi \cdot 5\right) + i\sin\left(\frac{5}{3}\pi \cdot 5\right)\right)$$

$$= \cos\left(\frac{25}{3}\pi\right) + i\sin\left(\frac{25}{3}\pi\right) =$$

$$\frac{25}{3}\pi = \frac{24}{3}\pi + \frac{1}{3}\pi = 8\pi + \frac{1}{3}\pi$$

$$= \frac{1}{3}\pi + k \cdot 2\pi, \quad k \in \mathbb{Z}$$

$$= \cos\left(\frac{1}{3}\pi\right) + i\sin\left(\frac{1}{3}\pi\right) = w^5$$

$$z = 2 \cdot \left(\cos\left(\frac{11}{6}\pi\right) + i\sin\left(\frac{11}{6}\pi\right)\right)$$

$$z^4 = 2^4 \cdot \left(\cos\left(\frac{11}{6}\pi \cdot 4\right) + i\sin\left(\frac{11}{6}\pi \cdot 4\right)\right) =$$

$$\frac{11}{6}\pi \cdot 4 = \frac{44}{6} \cdot \pi = \frac{42}{6}\pi + \frac{2}{6}\pi$$

$$= 7\pi + \frac{1}{3}\pi = 6\pi + \frac{4}{3}\pi = \frac{4}{3}\pi + k \cdot 2\pi, k \in \mathbb{Z}$$

$$= 16 \cdot \left(\cos\left(\frac{4}{3}\pi\right) + i\sin\left(\frac{4}{3}\pi\right)\right) = z^4$$

$$\frac{w^5}{z^4} = \frac{1 \cdot \left(\cos\left(\frac{1}{3}\pi\right) + i\sin\left(\frac{1}{3}\pi\right)\right)}{16 \cdot \left(\cos\left(\frac{4}{3}\pi\right) + i\sin\left(\frac{4}{3}\pi\right)\right)}$$

$$= \frac{1}{16} \cdot \left(\cos\left(\frac{1}{3}\pi - \frac{4}{3}\pi\right) + i\sin\left(\frac{1}{3}\pi - \frac{4}{3}\pi\right)\right)$$

$$= \frac{1}{16} \cdot \left(\cos(-\pi) + i\sin(-\pi)\right) =$$

$$-\pi + 2\pi = \pi$$

$$= \underline{\underline{\frac{1}{16} \cdot \left(\cos(\pi) + i\sin(\pi)\right)}}$$

$$w^5 - z^4 = \left(\cos\left(\frac{1}{3}\pi\right) + i\sin\left(\frac{1}{3}\pi\right)\right.$$
$$\left. - 16\cdot\left(\cos\left(\frac{4}{3}\pi\right) + i\sin\left(\frac{4}{3}\pi\right)\right)\right)$$

$$= \cos\left(\frac{1}{3}\pi\right) - 16\cdot\cos\left(\frac{4}{3}\pi\right)$$
$$+ \left(\sin\left(\frac{1}{3}\pi\right) - 16\cdot\sin\left(\frac{4}{3}\pi\right)\right) i$$

$$= \cos\left(\frac{1}{3}\pi\right) - 16\cdot\left(-\cos\left(\frac{1}{3}\pi\right)\right)$$
$$+ \left(\sin\left(\frac{1}{3}\pi\right) - 16\cdot\left(-\sin\left(\frac{1}{3}\pi\right)\right)\right) i$$

$$= 17\cdot\cos\left(\frac{1}{3}\pi\right) + 17\cdot\sin\left(\frac{1}{3}\pi\right) i$$
$$= \underline{\underline{17\cdot\left(\cos\left(\frac{1}{3}\pi\right) + i\sin\left(\frac{1}{3}\pi\right)\right)}}$$

b) $p(x) = x^5 - 3x^4 + 4x - 12$

ergibt sich aus Faktoren der Form
(irgendetwas mit $x \pm c$)

also Teiler von -12 betrachten:

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$

$p(1) = 1^5 - 3 \cdot 1 + 4 \cdot 1 - 12$
$= 1 - 3 + 4 - 12 \neq 0$

$p(-1) = -1 - 3 - 4 - 12 \neq 0$

$p(2) = 32 - 3 \cdot 16 + 8 - 12$
$= -16 + 8 - 12 \neq 0$

$p(-2) = (-2)^5 - 3 \cdot (-2)^4 + 4 \cdot (-2) - 12$
$= -32 - 3 \cdot 16 - 8 - 12 \neq 0$

$p(3) = 3^5 - 3 \cdot 3^4 + 4 \cdot 3 - 12 = 0$ ✓

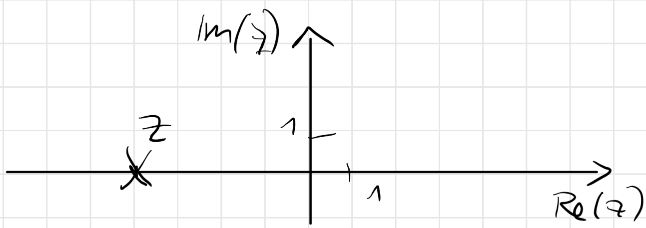
Polynomdivision

$$(x^5 - 3x^4 + 4x - 12) : (x-3) = x^4 + 4$$
$$-(x^5 - 3x^4)$$
$$0 \quad 4x - 12$$
$$-(4x - 12)$$
$$0$$

$$x^4 + 4 = 0$$
$$x^4 = -4$$

$\Rightarrow$ komplexes Wurzelze

$$w^4 = -4 = z$$



$$z = 4(\cos(\pi) + i\sin(\pi))$$

$$z = 4(\cos(\pi) + i\sin(\pi))$$

$$(w_4 =) w_0 = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right)$$

$$= 1 + i$$

$$w_1 = \sqrt[4]{4} \cdot \left(\cos\left(\frac{\pi}{4} + 1 \cdot \frac{2\pi}{4}\right)\right.$$

$$\left. + i\sin\left(\frac{\pi}{4} + 1 \cdot \frac{2\pi}{4}\right)\right)$$

$$= \sqrt{2} \cdot \left(\cos\left(\frac{3}{4}\pi\right) + i\sin\left(\frac{3}{4}\pi\right)\right)$$

$$= \sqrt{2} \cdot \left(-\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right)$$

$$= -1 + i$$

$$w_2 = \sqrt{2} \cdot \left(\cos\left(\frac{\pi}{4} + 2 \cdot \frac{2\pi}{4}\right)\right.$$

$$\left. + i\sin\left(\frac{\pi}{4} + 2 \cdot \frac{2\pi}{4}\right)\right)$$

$$w_2 = \sqrt{2} \cdot \left(\cos\left(\frac{5}{4}\pi\right) + i\sin\left(\frac{5}{4}\pi\right)\right)$$

$$= \sqrt{2} \cdot \left(-\frac{1}{\sqrt{2}} + i\left(-\frac{1}{\sqrt{2}}\right)\right)$$

$$= -1 - i$$

$$w_3 = \sqrt{2} \cdot \left(\cos\left(\frac{\pi}{4} + 3 \cdot \frac{2\pi}{4}\right) + i \sin\left(\frac{\pi}{4} + 3 \cdot \frac{2\pi}{4}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(\frac{7}{4}\pi\right) + i \sin\left(\frac{7}{4}\pi\right)\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i \cdot \left(-\frac{1}{\sqrt{2}}\right)\right)$$

$$= 1 - i$$

$$\Rightarrow p(x) = (x-3) \cdot (x-(1+i)) \cdot (x-(-1+i)) \cdot (x-(-1-i)) \cdot (x-(1-i))$$

$w_0 = w_4$

Antwort zu i)

ii) $p(x) = (x-3)\cdot(x-(1+i))\cdot(x-(-1+i))$
$\cdot\,(x-(-1-i))\cdot(x-(1-i))$

komplex konjugierte Paare

Allgemein

$(x-z)\cdot(x-\bar{z})$

$= (x-(a+bi))(x-(a-bi))$

$= (x-a-bi)\cdot(x-a+bi)$

$= ((x-a)-bi)((x-a)+bi)$

3. binomische Fo

$= (x-a)^2 + b^2$

$\overline{1+i} = 1-i \qquad \overline{-1+i} = -1-i$

$(x-(1+i))\cdot(x-(1-i)) = x^2-2x+2$

$(x-(-1+i))\cdot(x-(-1-i)) = x^2+2x+2$

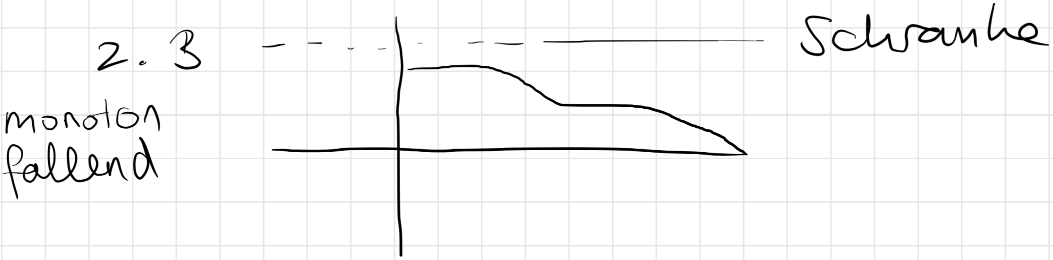
$p(x) = (x-3)\cdot(x^2-2x+2)(x^2+2x+2))$

Aufgabe V7:

Generell:

1. Das Wissen über Monotonie hilft bei den Schranken

2. 3



2. Monton fallend kann man auf verschiedene Arten zeigen

$a_{n+1} \geq a_n$

$a_{n+1} - a_n \geq 0$ oder $\frac{a_{n+1}}{a_n} \geq 1$

a) $a_{n+1} - a_n$

$$= \frac{4(n+1)-1}{(n+1)^2} - \frac{4n-1}{n^2}$$

$$= \frac{4n+4-1}{(n+1)^2} \cdot \frac{n^2}{n^2} - \frac{4n-1}{n^2} \cdot \frac{(n+1)^2}{(n+1)^2}$$

$$= \frac{(4n+3)n^2 - (4n-1)(n+1)^2}{\underbrace{(n+1)^2 \cdot n^2}_{>0}} \overset{?}{\lessgtr} 0$$

$$(4n+3)n^2 - (4n-1)(n+1)^2$$

$$= 4n^3 + 3n^2 - (4n-1)(n^2+2n+1)$$

$$= \cancel{4n^3} + 3n^2 - (\cancel{4n^3} - n^2 + 8n^2 - 2n + 4n - 1)$$

$$= +3n^2 + n^2 - 8n^2 + 2n - 4n + 1$$

$$= -5n^2 - 2n + 1$$

$$= -(\underbrace{5n^2+2n+1}_{\geq 1}) < 0$$

$\Rightarrow a_{n+1} \leq a_n$

$\Rightarrow$ monoton fallend

obere Schranke $a_1 = \frac{4 \cdot 1 - 1}{1} = 3$ (wegen Monotonie)

untere Schranke 0,

weil $\frac{4n-1}{n^2} \geq 0 \quad | \cdot n^2$

$4n - 1 \geq 0 \quad | +1$

$4n \geq 1$ ✓

b) $\frac{b_{n+1}}{b_n} = b_{n+1} : b_n = b_{n+1} \cdot \frac{1}{b_n}$

$$= \frac{(n+1)-1}{(n+1)+1} \cdot \frac{n-1}{n+1}$$

$$= \frac{n}{n+2} \cdot \frac{n+1}{n-1} \quad \text{für } n \neq 1$$

$$= \frac{n^2+n}{n^2+2n-n-2}$$

$$= \frac{n^2+n}{n^2+n-2} > 1 \quad \text{für } n \geq 2$$

$$\Rightarrow b_{n+1} > b_n$$

$b_1 = \frac{1-1}{1+1} = 0$, $b_2 = \frac{2-1}{2+1} = \frac{1}{3}$

$\Rightarrow$ streng monoton steigend für $n \in \mathbb{N}$

untere Schranke $0 = b_1$ wegen Monotonie)

oder weil $\frac{n-1}{n+1} \geq 0$

$$n-1 \geq 0$$

$$n \geq 0$$

obere Schranke 1

weil $\frac{n-1}{n+1} \leq 1$

$$n-1 \leq n+1$$

c) $\frac{c_{n+1}}{c_n}$

$$= \frac{\sqrt{(n+1)^2-(n+1)}}{\sqrt{n^2-n}} = \sqrt{\frac{(n+1)\cdot(n+1-1)}{n\cdot(n-1)}}$$

$$= \sqrt{\frac{n\cdot(n+1)}{n\cdot(n-1)}} \quad \sqrt{\frac{(n+1)}{(n-1)}} > 1$$

für $n > 1$

$c_1 = 0$, $c_2 = \sqrt{2^2-2} = \sqrt{2}$

$\Rightarrow$ streng monoton steigend für $n \in \mathbb{N}$

untere Schranke $c_1 = 0$ (wegen Monotonie)

obere Schranke $\sqrt{n^2-n} = \sqrt{n\cdot(n-1)}$

wächst immer weiter

d) $d_n = n^2 \cdot 5^{-n} = \frac{n^2}{5^n}$

$d_{n+1} - d_n$

$= \frac{(n+1)^2}{5^{n+1}} - \frac{n^2}{5^n}$

$= \frac{(n+1)^2 - 5n^2}{5^{n+1}} = \frac{n^2 + 2n + 1 - 5n^2}{5^{n+1}}$

$= \frac{\overbrace{-4n^2 + 2n + 1}^{<0}}{\underbrace{5^{n+1}}_{>0}} < 0$

$d_{n+1} - d_n < 0$

$d_{n+1} < d_n$

$\Rightarrow$ streng monoton fallend

obere Schranke $d_1 = \frac{1^2}{5^1} = \frac{1}{5}$ (wegen Monotonie)

untere Schranke $0 < \frac{\overbrace{n^2}^{>0}}{\underbrace{5^n}_{>0}}$