

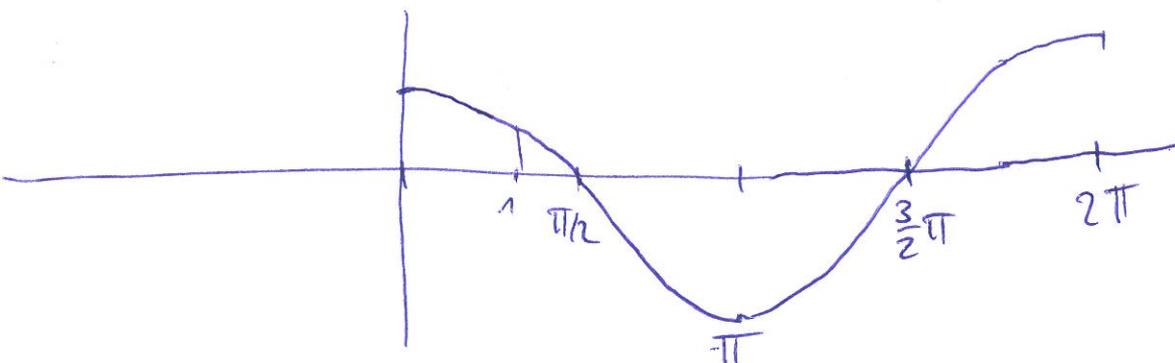
12.1

Subst. $u = e^x$

$$a) \int_0^{\ln(2\pi)} e^x \cos(e^x) dx$$

$$du = e^x dx$$

$$= \int_1^{2\pi} \cos(u) du = \sin(u) \Big|_1^{2\pi} = \sin(2\pi) - \sin(1) = -\sin(1)$$



$$b) \int_0^2 (x-3) \ln(x^2 - 6x + 9) dx$$

$$u = x - 3$$

$$du = dx$$

$$= \int_{-3}^{-1} u \ln(u^2) du$$

$$= \frac{1}{2} \int_{-3}^{-1} 2u \ln(u^2) du$$

$$= \frac{1}{2} \int_9^1 \ln(z) dz$$

$$= \frac{1}{2} \left(\ln(z)z \Big|_9^1 - \int_9^1 1 dz \right)$$

$$= \frac{1}{2} (\ln(z)z - z) \Big|_9^1$$

$$= 4 - 9 \ln(3)$$

$$z = u^2$$

$$dz = 2u du$$

NR:

$$Su'v' = uv - Su'v$$

Merke

Subst. $u = f(x)$

$$u' = \frac{du}{dx} = f'(x) \Rightarrow du = f'(x) dx$$

c) $\int_0^1 \frac{2x}{x^2+1} dx$ $u = x^2+1$
 $du = 2x dx$

$$= \frac{1}{2} \int_1^2 \frac{1}{u} du$$

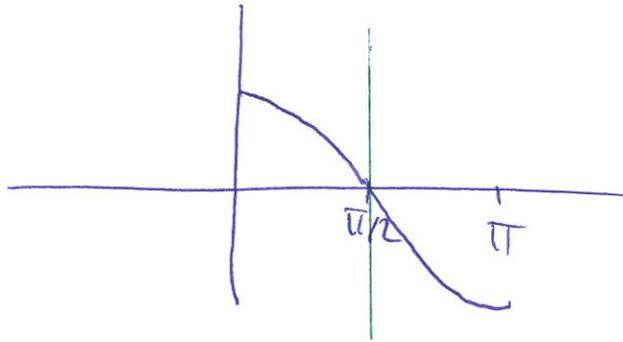
$$= \frac{1}{2} (\ln(2) - \ln(1)) = \frac{1}{2} \ln(2)$$

alternativ: $u = \sqrt{x^2+1}$
 $du = \frac{x}{\sqrt{x^2+1}} dx$

$$\begin{aligned} \int_0^1 \frac{x}{x^2+1} dx &= \int_0^1 \frac{1}{\sqrt{x^2+1}} \frac{x}{\sqrt{x^2+1}} dx \\ &= \int_1^{\sqrt{2}} \frac{1}{u} du = \ln(\sqrt{2}) \\ &= \frac{1}{2} \ln(2) \end{aligned}$$

$$\begin{aligned}
 d) \int_0^{\pi} \cos(x)^3 dx &= \int_0^{\pi} \cos(x) (1 - \sin(x)^2) dx \\
 &= \underbrace{\int_0^{\pi} \cos(x) dx}_{=0} - \int_0^{\pi} \cos(x) \sin(x)^2 dx
 \end{aligned}$$

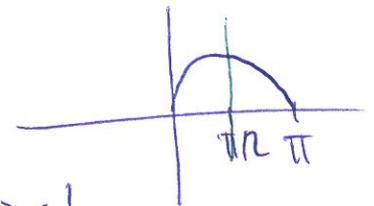
NR: Skizze



$$= - \int_0^{\pi} \cos(x) \sin(x)^2 dx$$

$$\begin{aligned}
 u &= \sin(x) \\
 du &= \cos(x) dx
 \end{aligned}$$

$$= - \int_0^0 u^2 du = 0$$



Achtung $\sin: [0, \pi] \rightarrow [-1, 1]$ nicht bijektiv!

Dh. Teilung des Integrationsgebiets ist notwendig:

$$\begin{aligned}
 &= - \int_0^{\pi} \cos(x) \sin(x)^2 dx = - \int_0^{\pi/2} \cos(x) \sin(x)^2 dx \\
 &\quad - \int_{\pi/2}^{\pi} \cos(x) \sin(x)^2 dx
 \end{aligned}$$

$$= - \int_0^1 u^2 du - \int_1^0 u^2 du = 0$$

12.2

$$(a) \quad \frac{x+1}{(x-1)(x^2+x+1)} \stackrel{!}{=} \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\begin{aligned} x+1 &\stackrel{!}{=} A(x^2+x+1) + (Bx+C)(x-1) \\ &= (A+B)x^2 + (A-B+C)x + (A-C) \cdot 1 \end{aligned}$$

$$\Rightarrow \quad A + B = 0$$

$$A - B + C = 1$$

$$A - C = 1$$

Damit durch $2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3$: $3A = 2 \Rightarrow A = 2/3$

und somit $B = -2/3$, $C = -1/3$

Insgesamt

$$\begin{aligned} \int \frac{x+1}{(x-1)(x^2+x+1)} dx &= \int \frac{2}{3} \frac{1}{x-1} dx - \frac{1}{3} \int \frac{2x+1}{x^2+x+1} dx \\ &= \frac{2}{3} \ln(|x-1|) + c - \frac{1}{3} \int \frac{2x+1}{x^2+x+1} dx, c \in \mathbb{R} \end{aligned}$$

NR

$$u = x^2+x+1$$

$$du = (2x+1) dx$$

$$= \frac{2}{3} \ln(|x-1|) + c - \frac{1}{3} \int \frac{1}{u} du, c \in \mathbb{R}$$

$$= \frac{2}{3} \ln(|x-1|) + c - \frac{1}{3} \ln(|x^2+x+1|), c \in \mathbb{R}$$

b) Sei $p(x) = x^3 + 3x^2 - x - 3$

$$p(1) = 1 + 3 - 1 - 3 = 0 \quad \checkmark$$

$$p(-1) = -1 + 3 + 1 - 3 = 0 \quad \checkmark$$

$$p(-3) = -27 + 27 + 3 - 3 = 0 \quad \checkmark$$

Somit

$$x^3 + 3x^2 - x - 3 = (x-1)(x+1)(x+3)$$

$$\int \frac{x+3}{x^3 + 3x^2 - x - 3} dx = \int \frac{x+3}{(x-1)(x+1)(x+3)} dx$$

$$= \int \frac{1}{(x-1)(x+1)} dx$$

$$\frac{1}{(x-1)(x+1)} \stackrel{!}{=} \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 \stackrel{!}{=} A(x+1) + B(x-1)$$

Gleichung soll für alle $x \in \mathbb{R}$ gelten. Insbesondere für $x=1$ und $x=-1$. Es folgt

$$x=1: \quad 1 \stackrel{!}{=} 2A + 0B, \text{ d.h. } A = \frac{1}{2}$$

$$x=-1: \quad 1 = 0A + (-2)B, \text{ d.h. } B = -\frac{1}{2}$$

Somit

$$\begin{aligned} \int \frac{1}{(x-1)(x+1)} dx &= \frac{1}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{1}{x+1} dx \\ &= \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + c, \quad c \in \mathbb{R} \end{aligned}$$

allg. Fall

$$\int \frac{ax+b}{x^2+cx+d}$$

, $a, b, c, d \in \mathbb{R}$

$$u = x^2+cx+d, \quad du = 2x+c$$

Schreibe

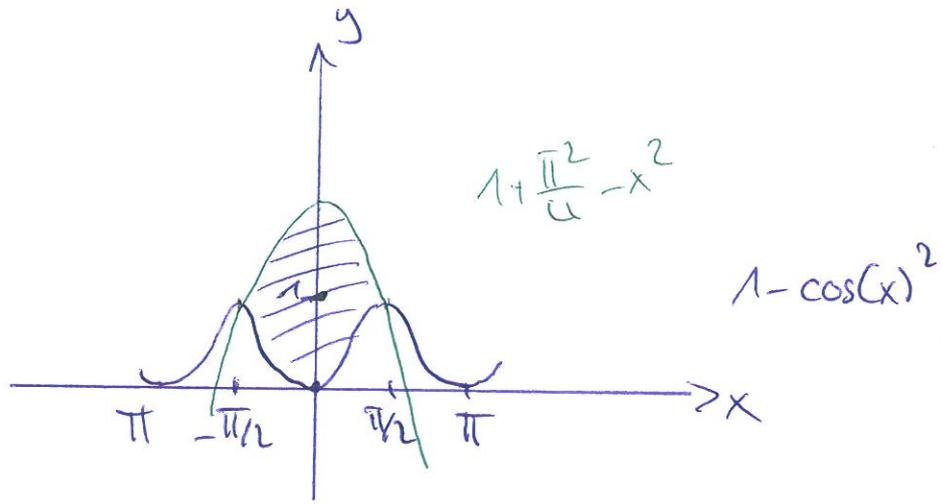
$$ax+b = ax + \frac{a}{2}c + (b - \frac{a}{2}c)$$

$$\frac{ax+b}{x^2+cx+d} = a \underbrace{\frac{x+\frac{c}{2}}{x^2+cx+d}}_{\substack{\rightarrow \text{subst. mit} \\ u = x^2+cx+d}} + \frac{b - \frac{a}{2}c}{\underbrace{x^2+cx+d}_{(x+\frac{c}{2})^2 + d - \frac{c^2}{4}}}$$

\rightarrow wird zu artan(...)
(Nachschlagen / Substitution)

12.3

Skizze:



$$\int_{-\pi/2}^{\pi/2} \left(1 + \frac{\pi^2}{4} - x^2 - (1 - \cos(x)^2) \right) dx$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{\pi^2}{4} - x^2 + \cos(x)^2 \right) dx$$

$$= \frac{\pi^3}{4} - \int_{-\pi/2}^{\pi/2} x^2 dx + \int_{-\pi/2}^{\pi/2} \cos(x)^2 dx$$

$$= \frac{\pi^3}{4} - \frac{1}{3} x^3 \Big|_{-\pi/2}^{\pi/2} + \left(\frac{x}{2} + \frac{\sin(2x)}{4} \right) \Big|_{-\pi/2}^{\pi/2} = \frac{1}{2} (1 + \cos(2x))$$

$$= \frac{\pi^3}{4} - \frac{1}{3} \frac{\pi^3}{4} + \frac{\pi}{4}$$

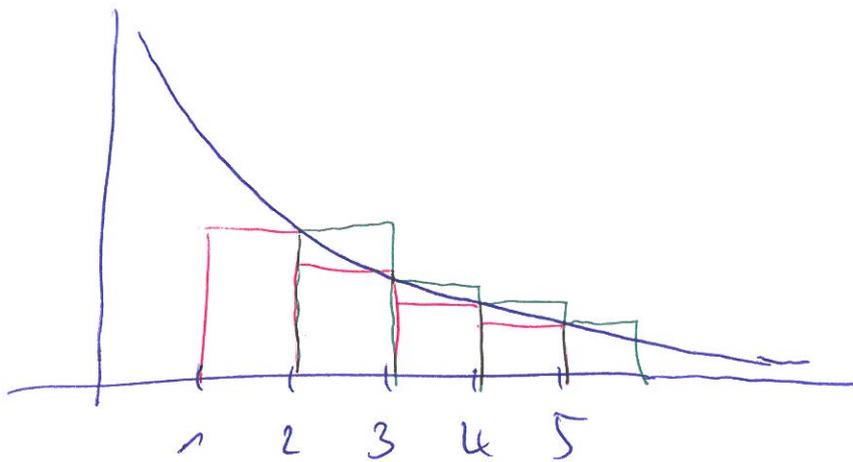
$$= \frac{1}{6} \pi^3 + \frac{\pi}{4}$$

12.4

Sei $f(x) = \frac{1}{x \ln(x)}$. Dann ist f monoton fallend und positiv!

und es gilt:

$$\sum_{k=2}^{\infty} f(k) < \infty \quad (\Leftrightarrow) \quad \int_2^{\infty} f(x) dx < \infty$$



a) Es ist: $\int_2^{\infty} \frac{1}{x \ln(x)} dx = \lim_{R \rightarrow \infty} \int_2^R \frac{1}{x \ln(x)} dx$

NR: $u = \ln(x)$, $du = \frac{1}{x} dx$

$= \lim_{R \rightarrow \infty} \int_{\ln(2)}^{\ln(R)} \frac{1}{u} du = \lim_{R \rightarrow \infty} \ln(\ln(R)) - \ln(\ln(2))$

b) $f(x) = \frac{1}{x \ln(x)^2}$ monoton fallend und positiv. Es gilt

$$\int_2^{\infty} \frac{1}{x \ln(x)^2} dx = \int_{\ln(2)}^{\infty} \frac{1}{u^2} du = \lim_{R \rightarrow \infty} \left(-\frac{1}{u}\right) \Big|_{\ln(2)}^R$$

$$= \frac{1}{\ln(2)} - \lim_{R \rightarrow \infty} \frac{1}{R} = \frac{1}{\ln(2)} < \infty.$$

Also konv. die Reihe!

12.5

Es gilt

$$I_n(x) = \int \sin(x)^n dx$$

$$= \int \sin(x)^{n-1} \cdot \sin(x) dx$$

$$= -\cos(x) \sin(x)^{n-1} + \int \sin(x)^{n-2} \cos(x)^2 \cdot (n-1) dx$$

$$= -\cos(x) \sin(x)^{n-1} + (n-1) \int \sin(x)^{n-2} - \sin(x)^n dx$$

$$= -\cos(x) \sin(x)^{n-1} + (n-1) \int \sin(x)^{n-2} dx - (n-1) \int \sin(x)^n dx$$

$$= -\cos(x) \sin(x)^{n-1} + (n-1) I_{n-2}(x) - (n-1) I_n(x).$$

Auflösen nach $I_n(x)$ liefert:

$$I_n(x) = -\frac{\cos(x) \sin(x)^{n-1}}{n} + \frac{n-1}{n} I_{n-2}(x).$$

Mit Verankerungen

$$I_0 = \int \sin(x)^0 dx = x + c, \quad c \in \mathbb{R}$$

bzw

$$I_1 = \int \sin(x)^1 dx = -\cos(x) + c, \quad c \in \mathbb{R}$$

je nachdem, ob n gerade oder ungerade ist.