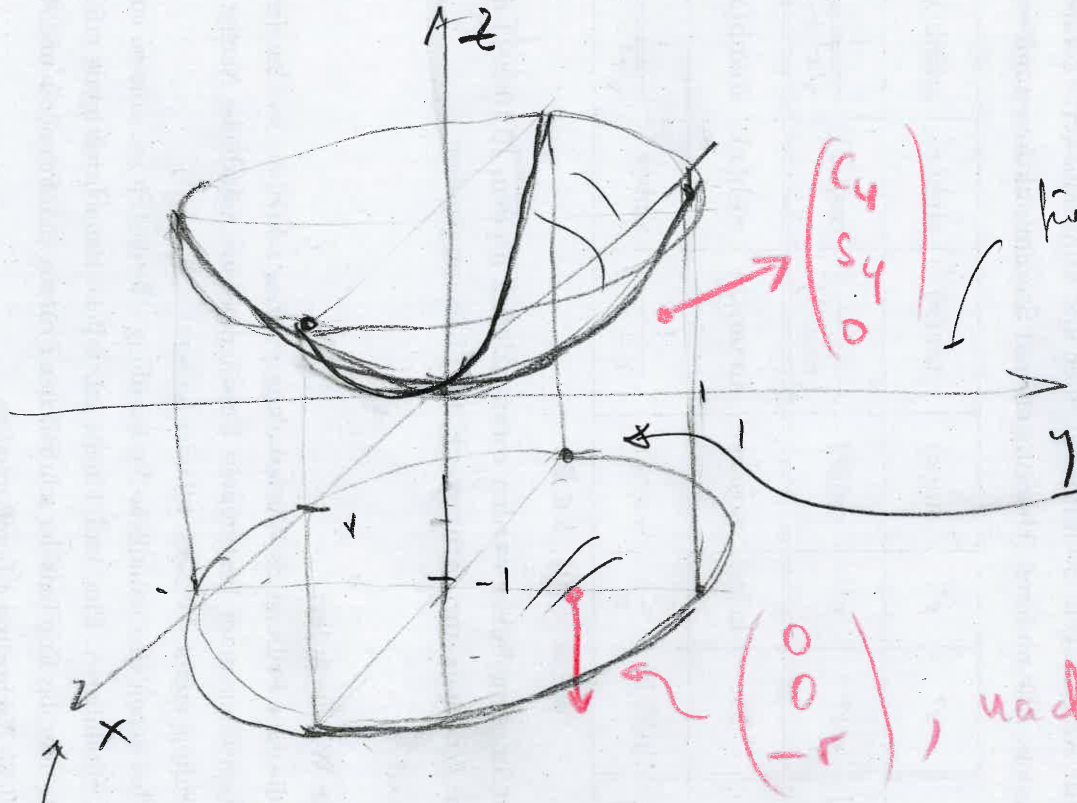
fr 24.11.23

A 4

(a) $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \underbrace{x^2+y^2 \le 1}_{\text{im Zylinder um } z\text{-Achse mit Radius } 1},\; \underbrace{-1}_{\text{„Boden"}} \le z \le \underbrace{x^2+y^2}_{\text{„gekrümmter Deckel"}} \right\}$



für $x = 0$: $z = y^2$

V, mit Oberfläche S

für $y = 0$: $z = x^2$

(b) $g = \begin{pmatrix} x y^2 z \\ -x^2 y z \\ z \end{pmatrix}$

Teilstücke von S: 1. Boden, 2. Deckel, 3. Mantel

1. Boden: $\phi_1(r,\varphi) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ -1 \end{pmatrix}$ } Standard-Polarkoordinaten

mit $r \in [0,1]$, $\varphi \in [0, 2\pi]$

Anteil A_1 am Ausfluss $A(g, S)$:

$$A_1 = \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} g(\phi_1(r,\varphi)) \cdot (\phi_{1,r}(r,\varphi) \times \phi_{1,\varphi}(r,\varphi)) \, dr \, d\varphi$$

$$g(\phi,(r,\varphi)) = g(r\underbrace{c_\varphi}_{\cos(\varphi)}, r\underbrace{s_\varphi}_{\sin(\varphi)}, -1)$$

$$= \begin{pmatrix} r c_\varphi \, r^2 s_\varphi^2 \, (-1) \\ -r^2 c_\varphi^2 \, r s_\varphi \, (-1) \\ -1 \end{pmatrix} = \begin{pmatrix} -r^3 c_\varphi s_\varphi^2 \\ r^3 c_\varphi^2 s_\varphi \\ -1 \end{pmatrix}$$

$$\phi_{1,r} = \begin{pmatrix} c_\varphi \\ s_\varphi \\ 0 \end{pmatrix}, \quad \phi_{1,\varphi} = \begin{pmatrix} -r s_\varphi \\ r c_\varphi \\ 0 \end{pmatrix},$$

$$\phi_{1,r} \times \phi_{1,\varphi} = \begin{pmatrix} c_\varphi \\ s_\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} -r s_\varphi \\ r c_\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r c_\varphi^2 - (-r s_\varphi^2) \end{pmatrix}$$

$$= -\begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

$r > 0 \rightsquigarrow$ zeigt nach V hinein

$\Rightarrow$ Vertauschen von r, φ

$$\Rightarrow A_1 = \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} \begin{pmatrix} -r^3 c_\varphi s_\varphi^2 \\ r^3 c_\varphi^2 s_\varphi \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} dr\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} r \; dr\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \left[\frac{r^2}{2} \right]_{r=0}^{r=1} d\varphi = \int_0^{2\pi} \frac{1}{2}\, d\varphi = \pi$$

2. Anteil A_2 am Ausfluß $A(g, S)$:

Deckel: $\Phi_2(r, \varphi) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ r^2 \end{pmatrix}$ } Standard-Polark.

$$A_2 = \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} g(\Phi_2(r, \varphi)) \bullet (\Phi_{2,r}(r,\varphi) \times \Phi_{2,\varphi}(r,\varphi))\; dr\, d\varphi$$

5.

$$g(\phi_2(r,\varphi)) = g(r c_\varphi, r s_\varphi, r^2) = \begin{pmatrix} r c_\varphi\, r^2 s_\varphi^2\, r^2 \\ -r^2 c_\varphi^2\, r s_\varphi\, r^2 \\ r^2 \end{pmatrix}$$

$$g = \begin{pmatrix} x y^2 z \\ -x^2 y z \\ z \end{pmatrix}$$

$$= \begin{pmatrix} r^5 c_\varphi s_\varphi^2 \\ -r^5 c_\varphi^2 s_\varphi \\ r^2 \end{pmatrix}$$

$$\phi_{2,r} = \begin{pmatrix} c_\varphi \\ s_\varphi \\ 2r \end{pmatrix}, \quad \phi_{2,\varphi} = \begin{pmatrix} -r s_\varphi \\ r c_\varphi \\ 0 \end{pmatrix},$$

$$\phi_{2,r} \times \phi_{2,\varphi} = \begin{pmatrix} c_\varphi \\ s_\varphi \\ 2r \end{pmatrix} \times \begin{pmatrix} -r s_\varphi \\ r c_\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -2r^2 c_\varphi \\ -2r^2 s_\varphi \\ r c_\varphi^2 - (-r s_\varphi^2) \end{pmatrix}$$

$$= \begin{pmatrix} -2r^2 c_\varphi \\ -2r^2 s_\varphi \\ r \end{pmatrix} \quad r > 0 : \left\{ \begin{array}{l} \text{zeigt} \\ \text{aus } V \text{ heraus} \end{array} \right.$$

$\Rightarrow$ ok

$$\Rightarrow \quad A_2 = \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} \begin{pmatrix} +r^5 c_\varphi s_\varphi^2 \\ -r^5 c_\varphi^2 s_\varphi \\ r^2 \end{pmatrix} \bullet \begin{pmatrix} -2r^2 c_\varphi \\ -2r^2 s_\varphi \\ r \end{pmatrix} dr\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} \cancel{-2r^7 c_\varphi^2 s_\varphi^2 + 2r^7 c_\varphi^2 s_\varphi^2} + r^3 \, dr\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \left[\frac{1}{4} r^4 \right]_{r=0}^{r=1} d\varphi = \int_{\varphi=0}^{\varphi=2\pi} \frac{1}{4}\, d\varphi$$

$$= \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$$

3. Anteil A_3 am Ausfluss $A(g, S)$:

Mantel: $\Phi_3(z, \varphi) = \begin{pmatrix} 1 \cdot \cos(\varphi) \\ 1 \cdot \sin(\varphi) \\ z \end{pmatrix}$

$z \in [-1, +1]$, $\varphi \in [0, 2\pi]$

$$A_3 = \int_{\varphi=0}^{\varphi=2\pi} \int_{z=-1}^{z=+1} g(\Phi_3(z,\varphi)) \bullet \left(\Phi_{3,z}(z,\varphi) \times \Phi_{3,\varphi}(z,\varphi)\right) dz\, d\varphi$$

$$g(\phi_3(z,\varphi)) = g(c_\varphi, s_\varphi, z) = \begin{pmatrix} c_\varphi s_\varphi^2 z \\ -c_\varphi^2 s_\varphi z \\ z \end{pmatrix}$$

$$g = \begin{pmatrix} x y^2 z \\ -x^2 y z \\ z \end{pmatrix}$$

$$\phi_{3,z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \phi_{3,\varphi} = \begin{pmatrix} -s_\varphi \\ c_\varphi \\ 0 \end{pmatrix}$$

$$\phi_{3,z} \times \phi_{3,\varphi} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -s_\varphi \\ c_\varphi \\ 0 \end{pmatrix} = - \begin{pmatrix} -c_\varphi \\ -s_\varphi \\ 0 \end{pmatrix}$$

muß nach außen zeigen!
Vertauschung von z, φ

$$\Rightarrow \quad A_3 = \int_{\varphi=0}^{\varphi=2\pi} \int_{z=-1}^{z=+1} \begin{pmatrix} c_\varphi s_\varphi^2 z \\ -c_\varphi^2 s_\varphi z \\ z \end{pmatrix} \cdot \begin{pmatrix} c_\varphi \\ s_\varphi \\ 0 \end{pmatrix} dz\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{z=-1}^{z=+1} \cancel{c_\varphi^2 s_\varphi^2 z - c_\varphi^2 s_\varphi^2 z} + 0 \quad dz\, d\varphi$$

$$= 0$$

Insgesamt:

$$A(g, S) = A_1 + A_2 + A_3$$

$$= \pi + \frac{\pi}{2} + 0$$

$$= \frac{3}{2} \pi .$$

(c) $\iiint_V \operatorname{div}(g)\, dx\, dy\, dz = \; ?$

V param. mit Zylinderkoordinaten:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\, c_\varphi \\ r\, s_\varphi \\ z \end{pmatrix} \qquad \begin{array}{l} r \in [0, 1] \\ \varphi \in [0, 2\pi] \\ z \in [-1, r^2] \end{array}$$

$$\operatorname{div} g = \operatorname{div} \begin{pmatrix} x y^2 z \\ -x^2 y z \\ z \end{pmatrix} = \frac{\partial}{\partial x} x y^2 z + \frac{\partial}{\partial y}(-x^2 y z) + \frac{\partial}{\partial z} z$$

$$= y^2 z - x^2 z + 1$$

$\Rightarrow \iiint_V \operatorname{div}(g)\, dx\, dy\, dz = \dots$

$\cdot\, r \leftarrow |\det(\text{Jacobi})|$ laut Skript r

$$\int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} \int_{z=-1}^{z=r^2} \left(r^2 s_\varphi^2 z - r^2 c_\varphi^2 z + 1 \right) dz\, dr\, d\varphi$$

$\cdot\, r$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} \left[\frac{1}{2} r^2 s_\varphi^2 z^2 - \frac{1}{2} r^2 c_\varphi^2 z^2 + z \right]_{z=-1}^{z=r^2} dr\, d\varphi$$

$\cdot\, r$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} \left(\left(\frac{1}{2} r^2 s_\varphi^2 r^4 - \frac{1}{2} r^2 c_\varphi^2 r^4 + r^2 \right) - \left(\frac{1}{2} r^2 s_\varphi^2 (-1)^2 - \frac{1}{2} r^2 c_\varphi^2 (-1)^2 + (-1) \right) \right) dr\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} \left(\frac{1}{2} r^6 \left(s_\varphi^2 - c_\varphi^2 \right) + r^2 - \frac{1}{2} r^2 \left(s_\varphi^2 - c_\varphi^2 \right) + 1 \right)$$

$\cdot\, r$ $dr\, d\varphi$

$= \ldots$

NR: $c_{2\varphi} = \cos(2\varphi) = \cos(\varphi+\varphi) = \cos(\varphi)^2 - \sin(\varphi)^2$

$$\Rightarrow s_\varphi^2 - c_\varphi^2 = -c_{2\varphi}$$

$$\ldots = \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} \left(-\frac{1}{2} r^6 c_{2\varphi} + r^2 + \frac{1}{2} r^2 c_{2\varphi} + 1\right) \cdot r \, dr \, d\varphi$$

$$= \int_{r=0}^{r=1} \int_{\varphi=0}^{\varphi=2\pi} \cancel{-\frac{1}{2} r^7 c_{2\varphi}} + r^3 + \cancel{\frac{1}{2} r^3 c_{2\varphi}} + r \, d\varphi \, dr$$

über 2 Perioden, $[0,\pi]$, $[\pi, 2\pi]$

$$= \int_{r=0}^{r=1} 2\pi \left(r^3 + r\right) dr$$

$$= 2\pi \left[\frac{1}{4} r^4 + \frac{1}{2} r^2\right]_{r=0}^{r=1} = 2\pi \left(\frac{1}{4} + \frac{1}{2}\right)$$

$$= 2\pi \cdot \frac{3}{4} = \frac{3}{2}\pi$$

(d) Es wird

$$A(g, S) \overset{(b)}{=} \frac{3}{2}\pi \overset{(c)}{=} \iiint \operatorname{div}(g)\,dx\,dy\,dz$$

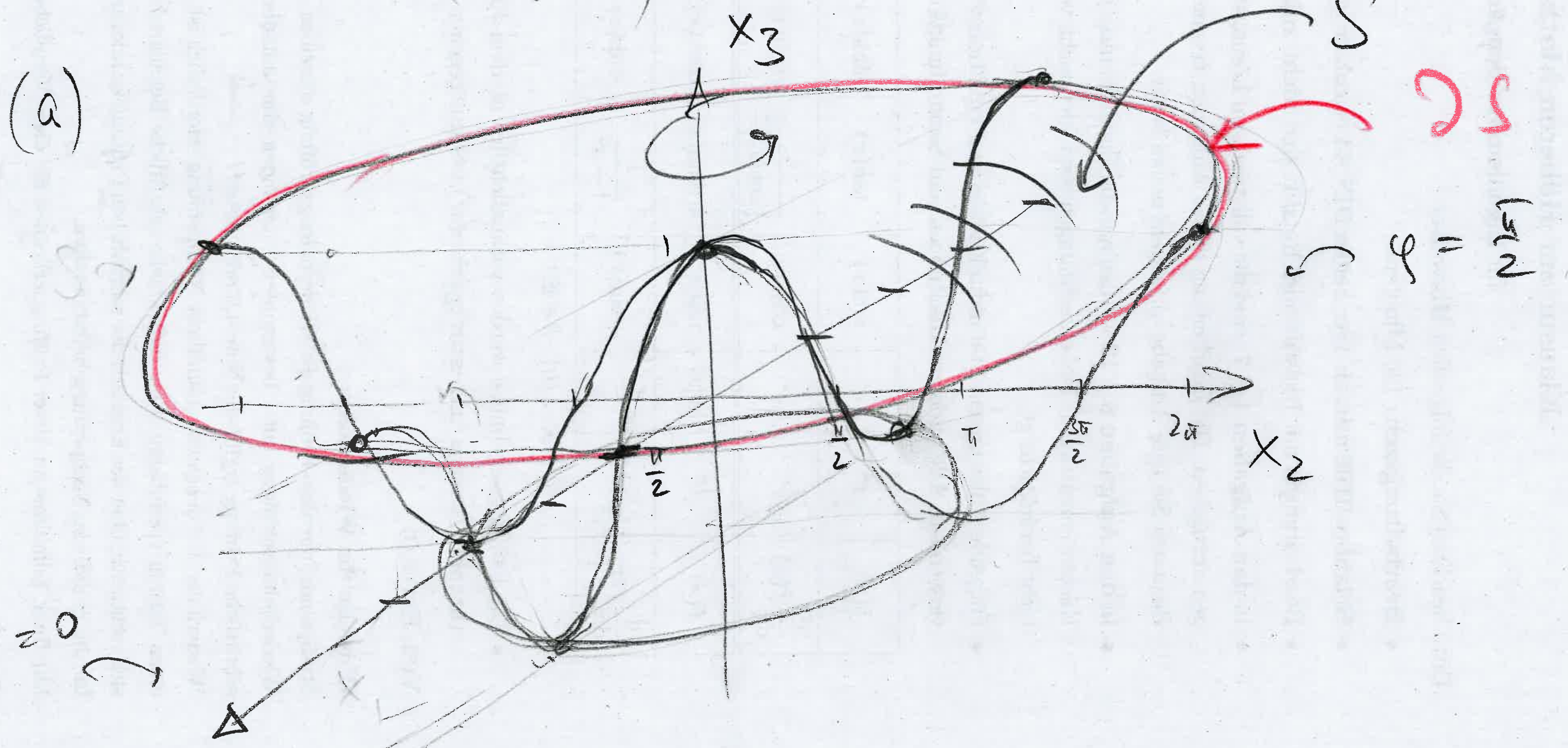
Dies bestätigt den Satz von Gauß im vorliegenden Fall.

Aufgabe 5 :

$$J = [0, 2\pi] \times (0, 2\pi]$$

$$\Phi(r, \varphi) = \begin{pmatrix} r\, c_\varphi \\ r\, s_\varphi \\ c_r \end{pmatrix}, \quad S = \Phi(J)$$

(a)

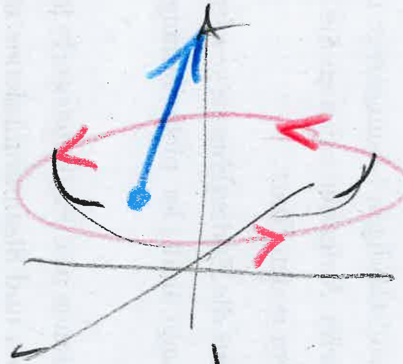


(c) $\int_{\partial S} g(x) \cdot dx = ?$

Nutzen Rechte-Hand-Regel!

$$C(t) = \begin{pmatrix} 2\pi \cos(t) \\ 2\pi \sin(t) \\ 1 \end{pmatrix}, \quad t \in [0, 2\pi]$$

Durchlaufungsrichtung!



$$\Phi_r \times \Phi_\varphi = \begin{pmatrix} c_\varphi \\ s_\varphi \\ -s_r \end{pmatrix} \times \begin{pmatrix} -r s_\varphi \\ r c_\varphi \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} r s_r c_\varphi \\ r s_r s_\varphi \\ r c_\varphi^2 + r s_\varphi^2 \end{pmatrix} = \begin{pmatrix} r s_r c_\varphi \\ r s_r s_\varphi \\ r \end{pmatrix}$$

$r > 0$

$\rightsquigarrow$ zeigt nach oben

Rechte - Hand - Regel paßt!

$$\Rightarrow \int_{\partial S} g(x) \cdot dx = \int_0^{2\pi} g(C(t)) \cdot C'(t)\, dt$$

Es ist $g(x_1, x_2, x_3) = \begin{pmatrix} -x_2 \\ 0 \\ 0 \end{pmatrix}$, also $g(C(t)) = g(2\pi c_t, 2\pi s_t, 1) = \begin{pmatrix} -2\pi s_t \\ 0 \\ 0 \end{pmatrix}$.

Also: $\int_{\partial S} g(x) \cdot dx = \int_0^{2\pi} \begin{pmatrix} -2\pi s_t \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2\pi s_t \\ 2\pi c_t \\ 0 \end{pmatrix} dt$

$$= \int_0^{2\pi} 4\pi^2 s_t^2\, dt = \int_0^{2\pi} 4\pi^2 \left(\frac{1}{2} - \frac{1}{2} c_{2t}\right) dt$$

$$= \left[4\pi^2 \left(\frac{t}{2} - \frac{1}{4} s_{2t} \right) \right]_0^{2\pi} = 4\pi^2 \cdot \frac{2\pi}{2} = 4\pi^3$$

(b) Wir wollen $\iint_S \mathrm{rot}(g) \cdot n \, dO$ berechnen. Dazu:

$\mathrm{rot}(g)(x_1, x_2, x_3) = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} \times \begin{pmatrix} -x_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, also $\mathrm{rot}(g)(\Phi(r, \varphi)) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\Phi_r(r, \varphi) \times \Phi_\varphi(r, \varphi) = \begin{pmatrix} r s_r c_\varphi \\ r s_r s_\varphi \\ r \end{pmatrix}$.

in (c) bereits bestimmt

Also:

$$\iint_S \operatorname{rot}(g) \cdot n \, dO = \iint_{\mathcal{F}} \operatorname{rot}(g)\left(\Phi(r,\varphi)\right) \cdot \left(\Phi_r(r,\varphi) \times \Phi_\varphi(r,\varphi)\right) dr \, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=2\pi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} r s_r c_\varphi \\ r s_r s_\varphi \\ r \end{pmatrix} dr \, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=2\pi} r \, dr \, d\varphi = \int_{\varphi=0}^{\varphi=2\pi} \left[\frac{1}{2} r^2\right]_0^{2\pi} d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \frac{1}{2}(2\pi)^2 \, d\varphi = 2\pi \cdot \frac{1}{2} \cdot (2\pi)^2 = 4\pi^3 .$$

(d) Es ist

$$\iint_S \operatorname{rot}(g) \cdot n \, dO \overset{(b)}{=} 4\pi^3 \overset{(c)}{=} \int_{\partial S} g(x) \cdot dx$$

Dies bestätigt den Satz von Stokes im vorliegenden Fall.