A9

(a) $\mathcal{L}(u(t)) = U(s) = \frac{1}{p(s)} = \frac{1}{s^3 - s^2 + 5s - 5}$

(b) $\mathcal{L}(e^{at} \cos(\omega t)) = \frac{s-a}{(s-a)^2 + \omega^2}$

$\mathcal{L}(e^{at} \sin(\omega t)) = \frac{\omega}{(s-a)^2 + \omega^2}$

$\mathcal{L}(e^{at}) = \frac{1}{s-a}$

~~PBZ:~~

Nullstellen von $p(s)$: $\lambda_1 = 1$

Polynomdivision

$$
\begin{array}{l}
(s^3 - s^2 + 5s - 5) : (s-1) = s^2 + 5 \\
\underline{s^3 - s^2} \\
\quad 0 + 5s - 5 \\
\quad \underline{\;\;\; 5s - 5} \\
\qquad\qquad 0
\end{array}
$$

Also $s^3 - s^2 + 5s - 5 = (s-1) \cdot (s^2 + 5)$

PBZ:

$$\frac{1}{s^3 - s^2 + 5s - 5} = \frac{A}{s-1} + \frac{Bs + C}{s^2 + 5}$$

$$= \frac{A(s^2+5)}{(s-1)(s^2+5)} + \frac{(Bs+C)\cdot(s-1)}{(s-1)\cdot(s^2+5)}$$

$$= \frac{(A+B)\cdot s^2 + (-B+C)s + (5A - C)}{(s-1)(s^2+5)}$$

$$\Leftrightarrow \left.\begin{array}{r} A + B = 0 \\ -B + C = 0 \\ 5A - C = 1 \end{array}\right\} \Leftrightarrow \left.\begin{array}{r} A = -B \\ B = C \\ 6A = 1 \end{array}\right\} \Leftrightarrow \begin{array}{l} B = -\frac{1}{6} \\ C = -\frac{1}{6} \\ A = \frac{1}{6} \end{array}$$

Also: $\frac{1}{s^3 - s^2 + 5s - 5} = \frac{1}{6} \frac{1}{s-1} - \frac{1}{6} \frac{s+1}{s^2+5}$

$$\mathcal{L}^{-1}\left(\underbrace{\frac{1}{6}\frac{1}{s-1}} - \frac{1}{6}\frac{s}{s^2+5} - \frac{1}{6}\frac{1}{s^2+5}\right)$$

$$= \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{6}\mathcal{L}^{-1}\left(\frac{s}{s^2+5}\right) - \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s^2+5}\right)$$

$$= \frac{1}{6}e^{1\cdot t} - \frac{1}{6}e^{0\cdot t}\cdot\cos(\sqrt{5}t) - \frac{1}{6\sqrt{5}}\cdot e^{0\cdot t}\cdot\sin(\sqrt{5}t)$$

$$= u(t)$$

$\frac{s-0}{(s^2+5)}$

$a=0$

$\boxed{\omega^2=5}$

$\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{s^2+5}$

$a=0$

$\omega^2=5$

NS von s^2+5:

$0 \pm \sqrt{5}\,i$

FS: $f_1(t)=e^{0t}\cos(\sqrt{5}t)$

$f_2(t)=e^{0t}\sin(\sqrt{5}t)$

(c)

aus VL: $f(t) = (u*g)(t)$ $\qquad g(t)=3$

$$(u*g)(t) = \int_0^t u(t-\tau)\cdot g(\tau)\,d\tau$$

$$= (g*u)(t) = \int_0^t g(t-\tau)\cdot u(\tau)\,d\tau$$

$$= \int_0^t 3\cdot\left(\frac{1}{6}e^{\tau} - \frac{1}{6}\cos(\sqrt{5}\tau) - \frac{1}{6\sqrt{5}}\sin(\sqrt{5}\tau)\right)d\tau$$

$$= \left[\frac{1}{2}e^{\tau} - \frac{3}{6\sqrt{5}}\sin(\sqrt{5}\tau) + \frac{3}{6\cdot 5}\cos(\sqrt{5}\tau)\right]_0^t$$

$$= \frac{1}{2}e^t - \frac{1}{2\sqrt{5}}\sin(\sqrt{5}t) + \frac{1}{10}\cos(\sqrt{5}t) - \frac{1}{2} - \frac{1}{10}$$

$$= -\frac{6}{10} = -\frac{3}{5}$$

$$= \frac{1}{2}e^t - \frac{\sqrt{5}}{10}\sin(\sqrt{5}t) + \frac{1}{10}\cos(\sqrt{5}t) - \frac{3}{5}$$

A10

$$y' - \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}_{A} y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) EW von A: $\lambda_1 = 2$, $\lambda_2 = 2$

EV zu λ_1:

$$(A - \lambda E_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$rg\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 = 2 - 1$$

Eigenraum zu $\lambda = 2$:

$E(2) = \left\{ a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}, a \in \mathbb{R} \right\}$

Falls 2 l.u. EV v_1, v_2 gefunden:

Fundamentalsystem: $v_1 \cdot e^{2x}$, $v_2 \cdot e^{2x}$

(b) $B = (A - 2E_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$B^0 v = v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $B^1 v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$B^2 v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(c) $B^k v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ und $B^{k-1} v \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ~~Bei~~

Bei uns: $k = 2$

Fundamentalsystem:

$$f_{[1]}(x) = e^{2x} \left(\frac{x^0}{0!} B^{k-1} v \right) = e^{2x} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f_{[2]}(x) = e^{2x} \left(\frac{x^1}{1!} B^{k-1} v + \frac{x^0}{0!} B^{k-2} v \right)$$

$$= e^{2x} \left(x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Allg Lösung des homogenen Systems:

$$y_h(x) = c_1 f_{[1]}(x) + c_2 \cdot f_{[2]}(x) = e^{2x} \begin{pmatrix} c_1 + c_2 \cdot x \\ c_2 \end{pmatrix}$$

(d) partikuläre Lösung: Variation der Konstanten

$$\begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = W_{sys}^{-1} \cdot b(x) = W_{sys}^{-1} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$W_{sys}(x) = \begin{pmatrix} f_{[1]} & f_{[2]} \end{pmatrix} = \begin{pmatrix} e^{2x} & xe^{2x} \\ 0 & e^{2x} \end{pmatrix}$$

$$W_{sys}^{-1}(x) = \frac{1}{e^{4x}} \cdot \begin{pmatrix} e^{2x} & -xe^{2x} \\ 0 & e^{2x} \end{pmatrix} = e^{-2x} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$$

Für 3x3

$$W_{sys}^{-1}(x) = W_{sys}^{-1}(0) \cdot W_{sys}(-x) \cdot W_{sys}^{-1}(0)$$

$$\begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = e^{-2x} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{-2x} \begin{pmatrix} -x \\ 1 \end{pmatrix}$$

$$c_1(x) = \int e^{-2x} \cdot (-x)\, dx = \left[-\tfrac{1}{2} e^{-2x} (-x) \right] - \int \tfrac{1}{2} e^{-2x} dx$$

$$= \left[\tfrac{1}{2} xe^{-2x} + \tfrac{1}{4} e^{-2x} \right]$$

$$c_2(x) = \int e^{-2x}\, dx = \left[-\tfrac{1}{2} e^{-2x} \right]$$

$$y_p(x) = c_1(x) \cdot f_{[1]}(x) + c_2(x) \cdot f_{[2]}(x)$$

$$= e^{2x} \cdot \begin{pmatrix} \cancel{\tfrac{1}{2} xe^{-2x}} + \tfrac{1}{4} e^{-2x} \\ 0 \end{pmatrix} + e^{2x} \cdot \begin{pmatrix} \cancel{x \cdot \left(-\tfrac{1}{2} e^{-2x}\right)} \\ 1 \cdot \left(-\tfrac{1}{2} e^{-2x}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \tfrac{1}{4} \\ -\tfrac{1}{2} \end{pmatrix}$$

Probe: $\quad y_p(x) = \begin{pmatrix} \tfrac{1}{4} \\ -\tfrac{1}{2} \end{pmatrix} \qquad y_p'(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1/4 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad ✓$$

$$y(x) = y_p(x) + y_h(x) = \begin{pmatrix} 1/4 \\ -1/2 \end{pmatrix} + e^{2x}\begin{pmatrix} c_1 + c_2 x \\ c_2 \end{pmatrix}$$

$$y(0) = \begin{pmatrix} 1/4 \\ -1/2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \left.\begin{matrix} 1/4 + c_1 = 1 \\ -\frac{1}{2} + c_2 = 1 \end{matrix}\right\} \Leftrightarrow \left.\begin{matrix} c_1 = 3/4 \\ c_2 = \frac{3}{2} \end{matrix}\right\}$$

Also

$$y(x) = \begin{pmatrix} 1/4 \\ -1/2 \end{pmatrix} + e^{2x}\begin{pmatrix} 3/4 + \frac{3}{2}x \\ \frac{3}{2} \end{pmatrix}$$

A11

$$y' = \underbrace{\begin{pmatrix} 3 & -1 & 3 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{A} y$$

EW von A:

$$\det\left(\begin{pmatrix} (3-\lambda) & -1 & 3 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix}\right) = (3-\lambda)\cdot \det\left(\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}\right)$$

$$= (3-\lambda)(\lambda^2+1) \quad \Rightarrow \lambda_1 = 3 \;, \; \lambda_{2,3} = \pm i$$

EV zu $\lambda_1 = 3$: $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad f_{[1]}(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot e^{3x}$

zu komplexen EW: $\lambda = \underbrace{\alpha}_{=0} + i \underbrace{\beta}_{=1}$ und EV $w = u + iv$

aus VL: $f_{[2]}(x) = e^{\alpha x}(\cos(\beta x)) \cdot u - \sin(\beta x)\cdot v)$

$f_{[3]}(x) = e^{\alpha x}(\sin(\beta x)\cdot u + \cos(\beta x) \cdot v)$

EV von $\lambda = i$:

$$\begin{pmatrix} 3-i & -1 & 3 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{pmatrix} \xrightarrow{\cdot(-i)} \begin{pmatrix} 3-i & -1 & 3 \\ 0 & -i & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\cdot i} \begin{pmatrix} 3-i & 0 & 3-i \\ 0 & -i & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} 1\cdot\frac{1}{3-i} \\ 1\cdot i \\ \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow w = \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{u} + i \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v}$$

$$f_{[2]}(x) = e^{0x}\cdot\left(\cos(1\cdot x)\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \sin(1\cdot x)\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -\cos(x) \\ -\sin(x) \\ \cos(x) \end{pmatrix}$$

$$f_{[3]}(x) = \sin(x)\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \cos(x)\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin(x) \\ \cos(x) \\ \sin(x) \end{pmatrix}$$

Allgemeine homogene Lösung:

$$y(x) = y_h(x) = c_1 \cdot f_{[1]}(x) + c_2 \cdot f_{[2]}(x) + c_3 \cdot f_{[3]}(x) \qquad c_1, c_2, c_3 \in \mathbb{R}$$

$$= \begin{pmatrix} c_1 e^{3x} - c_2\cos(x) - c_3\sin(x) \\ -c_2\sin(x) + c_3\cos(x) \\ c_2\cos(x) + c_3\sin(x) \end{pmatrix}$$

$\cos(\pi) = -1$
$\sin(\pi) = 0$

$$y(\pi) = \begin{pmatrix} c_1 e^{3\pi} + c_2 \\ -c_3 \\ -c_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \left.\begin{array}{l} c_1 e^{3\pi} + c_2 = 0 \\ -c_3 = 1 \\ -c_2 = 1 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} c_1 e^{3\pi} - 1 = 0 \\ c_2 = -1 \\ c_3 = -1 \end{array}\right\} \Leftrightarrow \begin{array}{l} c_1 = e^{-3\pi} \\ c_2 = -1 \\ c_3 = -1 \end{array}$$

$$y(x) = \begin{pmatrix} e^{-3\pi} \cdot e^{3x} + \cos(x) + \sin(x) \\ \sin(x) - \cos(x) \\ -\cos(x) - \sin(x) \end{pmatrix}$$