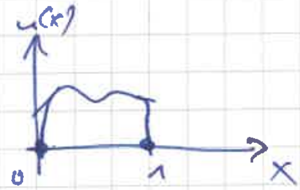
VÜ 6

A15

$$u_t = \underbrace{2}_{a^2} u_{xx} \qquad u(0,t) = u(l,t) = 0 \qquad t \geq 0$$



$$u(x,0) = \cancel{\sin(x)} \;\overbrace{\sin(\pi x) + \sin(2\pi x)}^{=f}$$

a) $$u(x,t) = \sum_{n=1}^{\infty} b_n \cdot u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l} x\right) e^{-\frac{a^2 n^2 \pi^2}{l^2} \cdot t}$$

$a^2 = 2$, $l = 1$
$$= \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-n^2\pi^2 t \cdot 2}$$

$$b_n = \frac{2}{l} \int_0^l f(\xi) \sin\left(\frac{n\pi}{l}\xi\right) d\xi \qquad f \sim \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{l} x\right)$$

$$= 2 \int_0^1 \left(\sin(\pi\xi) + \sin(2\pi\xi)\right) \cdot \sin(n\pi\xi)\, d\xi$$

$$\boxed{\begin{aligned} \pi\xi &= \tilde{\xi} \\ d\xi &= \tfrac{1}{\pi} d\tilde{\xi} \end{aligned}}$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\sin(\tilde{\xi}) + \sin(2\tilde{\xi})\right) \cdot \sin(n\tilde{\xi})\, d\tilde{\xi}$$

$$= \frac{2}{\pi} \left[\underbrace{\int_0^{\pi} \sin(\tilde{\xi}) \cdot \sin(n\tilde{\xi})\, d\tilde{\xi}}_{= \begin{cases} \frac{\pi}{2} & \text{für } n=1 \\ 0 & \text{sonst} \end{cases}} + \underbrace{\int_0^{\pi} \sin(2\tilde{\xi}) \sin(n\tilde{\xi})\, d\tilde{\xi}}_{= \begin{cases} \frac{\pi}{2} & \text{für } n=2 \\ 0 & \text{sonst} \end{cases}} \right]$$

$$\Rightarrow \quad b_1 = 1, \quad b_2 = 1, \quad b_n = 0 \text{ für } n \geq 3$$

$$f(x) = \underbrace{\sin(\pi x)}_{\Rightarrow b_1 = 1} + \underbrace{\sin(2\pi x)}_{\Rightarrow b_2 = 1} \sim \sum_{n=1}^{\infty} b_n \cdot \sin(n\pi x)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-n^2\pi^2 t \cdot 2} = \sin(\pi x) \cdot e^{-\pi^2 t \cdot 2} + \sin(2\pi x) \cdot e^{-4\pi^2 t \cdot 2}$$

$$= \sin(\pi x)\, e^{-2\pi^2 t} + \sin(2\pi x)\, e^{-8\pi^2 t}$$

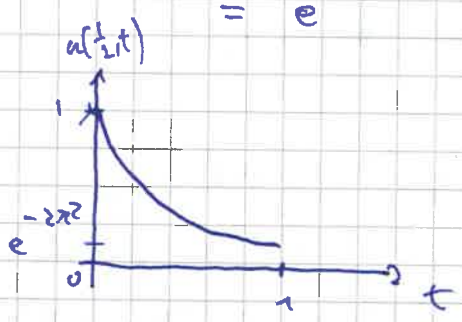
$$\int_{-\pi}^{\pi} \sin(nx)\sin(kx)\, dx = \begin{cases} \pi & n = k \neq 0 \\ 0 & \text{sonst} \end{cases}$$

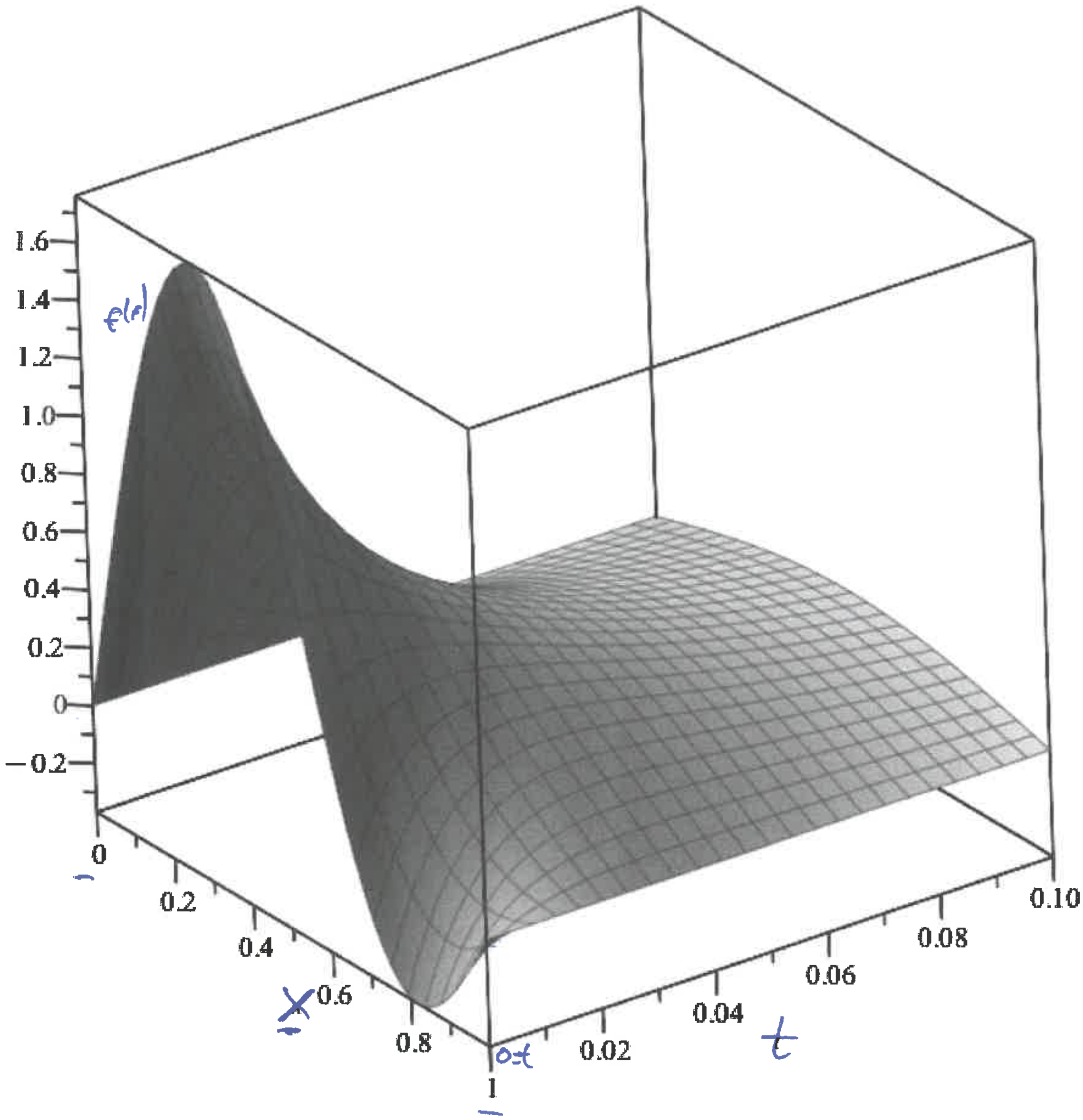
$$= \tfrac{2}{2}\, 2 \cdot \int_0^{\pi} \text{— " —}$$

(b) $u(x,t) = \sin(\pi x)\, e^{-2\pi^2 t} + \sin(2\pi x)\, e^{-8\pi^2 t}$

$$u\left(\tfrac{1}{2},t\right) = \underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{=1} e^{-2\pi^2 t} + \underbrace{\sin(\pi)}_{=0} e^{-8\pi^2 t}$$

$$= e^{-2\pi^2 t}$$





1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
−0.2
0
0.2
0.4
0.6
0.8
1
X
0.02
0.04
0.06
0.08
0.10
t

VÜ 6

A16

$$u_t = u_{xx} + \underbrace{1}_{r(x,t)} \qquad u(0,t) = u(\pi,t) = 0 \quad t \geq 0$$
$$u(x,0) = 0 \qquad x \in (0,\pi)$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L} x\right) e^{-\frac{a^2 n^2 \pi^2}{L^2} t}$$

$L = \pi$, $a^2 = 1$

$$b_n'(t) = \tilde{b}_n(t) e^{\frac{a^2 n^2 \pi^2}{L^2} t} = \tilde{b}_n(t) e^{n^2 t}$$

$$\tilde{b}_n(t) = \frac{2}{L} \int_0^L r(x,t) \sin\left(\frac{n\pi}{L} x\right) dx \qquad r(x,t) \sim \sum_{n=1}^{\infty} \tilde{b}_n(t) \sin\left(\frac{n\pi}{L} x\right)$$

$$= \frac{2}{\pi} \int_0^{\pi} \cancel{\sin(n\pi)} \sin(nx)\, dx$$

$$= \frac{2}{\pi} \left[-\cos(nx) \cdot \frac{1}{n} \right]_0^{\pi}$$

$$= \frac{2}{n\pi} \left(-\cos(n\pi) + 1 \right) = \frac{2}{n\pi} \left(-(-1)^n + 1 \right) = \begin{cases} 0 & n \text{ gerade} \\ \frac{4}{n\pi} & n \text{ ungerade} \end{cases}$$

$$\Rightarrow b_n'(t) = \frac{2}{n\pi} \cdot \left((-1)^{n+1} + 1 \right) \cdot e^{n^2 t}$$

$$b_n(t) = \int \frac{2}{n\pi} \left((-1)^{n+1} + 1 \right) e^{n^2 t} dt$$

$$= \frac{2}{n^3 \pi} \left((-1)^{n+1} + 1 \right) e^{n^2 t} + c_n$$

Also

$$u(x,t) = \sum_{n=1}^{\infty} \underbrace{\left(\frac{2}{n^3 \pi} \left((-1)^{n+1} + 1 \right) e^{n^2 t} + c_n \right)}_{b_n(t)} \sin(nx) e^{-n^2 t}$$

$$u(x,0) = f(x) = 0$$

$$\sum_{n=1}^{\infty} b_n(0) \sin(nx) \sim f(x)$$

Also müssen wir $f(x)$ in eine Fourierreihe entwickeln

$$f(x) \sim \sum_{n=1}^{\infty} d_n \sin(nx)$$

Fourierreihe von $f(x)$: $\quad f(x) = 0 \sim \sum_{n=1}^{\infty} d_n \sin(nx)$

mit $d_n = 0, \; n \geq 1$

Also:

$$d_n \overset{!}{=} b_n(0)$$

$$0 = \frac{2}{n^3\pi}\left((-1)^{n+1}+1\right) e^{n^2 \cdot 0} + c_n$$

$$c_n = -\frac{2}{n^3\pi}\left((-1)^{n+1}+1\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \underbrace{\left(\overbrace{\frac{2}{n^3\pi}\left((-1)^{n+1}+1\right) e^{n^2 t}}^{u_p} + \overbrace{\underbrace{\left(-\frac{2}{n^3\pi}\left((-1)^{n+1}+1\right)\right)}_{c_n} \sin(nx) e^{-n^2 t}}^{u_h} \right)}_{b_n(t)}$$

$$= \sum_{n=1}^{\infty} \underbrace{\left(\frac{2}{n^3\pi}\left((-1)^{n+1}+1\right)\left(e^{n^2 t} - 1\right)\right)}_{b_n(t)} \sin(nx) e^{-n^2 t}$$

$$= \sum_{n=1}^{\infty} \frac{2}{n^3\pi}\left((-1)^{n+1}+1\right)\left(1 - e^{-n^2 t}\right) \sin(nx)$$

$\Rightarrow$ für $u_p = \sum_{n=1}^{\infty} \left(\frac{2}{n^3\pi}\left((-1)^{n+1}+1\right) \cancel{e^{n^2 t}} \right) \cdot \sin(nx) \cancel{e^{-n^2 t}}$

$$u_h(x,t) = \sum_{n=1}^{\infty} b_n \cdot \sin(nx) e^{-n^2 t}$$

Üü 6

A 17

$$u_{tt} = \underbrace{4}_{c^2} u_{xx} \qquad u(x,0) = \cos(x) \qquad x \in \mathbb{R}$$

(a) Satz v. d'Alambert

$$u(x,t) = F(x-ct) + G(x+ct)$$
$$= F(x-2t) + G(x+2t)$$

$$F(x) + G(x) = u(x,0) \overset{!}{=} \cos(x)$$

z.B.: $F(x) = \cos(x)$, $G(x) = 0$

$$\underline{\underline{u(x,t) = \cos(x-2t)}}$$

Prüfe ob $u(x,t)$ die Wellengleichung erfüllt:

$$u_x(x,t) = \frac{\partial}{\partial x} u(x,t) = -\sin(x-2t)$$

$$u_{xx}(x,t) = \frac{\partial}{\partial x} u_x(x,t) = -\cos(x-2t)$$

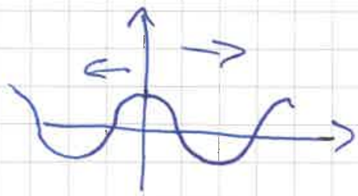
$$u_t(x,t) = \frac{\partial}{\partial t} u(x,t) = -\sin(x-2t) \cdot (-2) = 2\sin(x-2t)$$

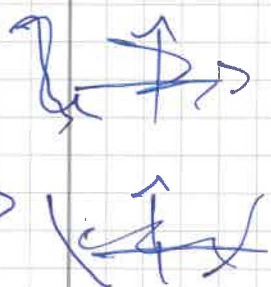
$$u_{tt}(x,t) = -4\cos(x-2t)$$

$$u_{tt} = -4\cos(x-2t) = 4 \cdot u_{xx} \qquad ✓$$

b) $F(x) = 0$, $G(x) = \cos(x)$

$$u(x,t) = \cos(x+2t) \neq \cos(x-2t)$$





$F(x) = \frac{1}{2}\cos(x)$, $G(x) = \frac{1}{2}\cos(x)$

$$u(x,t) = \frac{1}{2}\cos(x-2t) + \frac{1}{2}\cos(x+2t)$$

$$\left.\begin{aligned} F(x) &= \cos(x) + e^{x^2} \\ G(x) &= -e^{+x^2} \end{aligned}\right\} \Rightarrow u(x,t) = \cos(x-2t) + e^{(x-2t)^2} - e^{(x+2t)^2}$$