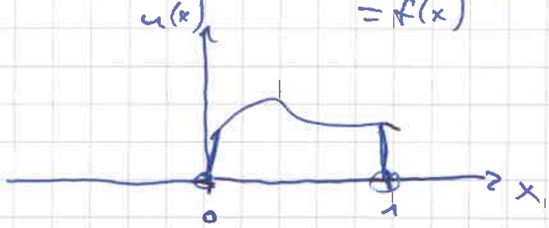
# VÜ 6

A15

$$\underbrace{u_t = \underbrace{2}_{=a^2} u_{xx} + \underbrace{r(x,t)}_{=0}}\qquad u(0,t) = u(\underbrace{1}_{=L},t) = 0\,, \qquad t \geq 0$$

$$u(x,0) = \underbrace{\sin(\pi x) + \sin(2\pi x)}_{=f(x)} \qquad x \in (0,\underbrace{1}_{=L})$$




(a) $$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right) e^{-\frac{a^2 n^2 \pi^2}{L^2} \cdot t}$$

$$u_t = a^2 u_{xx} \quad \text{für} \quad x \in [0,L]$$

$L = 1,\ a^2 = 2$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cdot \sin(n\pi x)\, e^{-2n^2\pi^2 \cdot t}$$

$$b_n = \frac{2}{L} \int_0^L f(\xi) \sin\left(\frac{n\pi}{L}\xi\right) d\xi$$

$$= 2 \cdot \int_0^1 \left(\sin(\pi\xi) + \sin(2\pi\xi)\right) \cdot \sin(n\pi\xi)\, d\xi$$

$$\boxed{\begin{aligned} \pi\xi &= \tilde{\xi} \\ d\xi &= \frac{1}{\pi} d\tilde{\xi} \end{aligned}}$$

$$= 2 \cdot \frac{1}{\pi} \int_0^{\pi} \left(\sin(\tilde{\xi}) + \sin(2\tilde{\xi})\right) \cdot \sin(n\tilde{\xi})\, d\tilde{\xi}$$

$$= \frac{2}{\pi}\left( \underbrace{\int_0^{\pi} \sin(\tilde{\xi}) \sin(n\tilde{\xi})\, d\tilde{\xi}}_{= \begin{cases} \pi/2 & n=1 \\ 0 & \text{sonst} \end{cases}} + \underbrace{\int_0^{\pi} \sin(2\tilde{\xi}) \sin(n\tilde{\xi})\, d\tilde{\xi}}_{= \begin{cases} \pi/2 & n=2 \\ 0 & \text{sonst} \end{cases}} \right)$$

Orth. relation

$$\int_{-\pi}^{\pi} \sin(nx)\sin(kx)\,dx = \begin{cases} \pi & n=k\neq 0 \\ 0 & \text{sonst} \end{cases}$$

$$= \begin{cases} 1 & \text{für } n=1 \text{ oder } n=2 \\ 0 & \text{sonst} \end{cases}$$

$\Rightarrow \quad b_1 = b_2 = 1$

$b_n = 0 \qquad n \geq 3$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \cdot \sin(n\pi x)$$

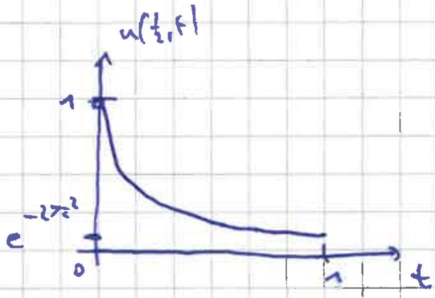
$$f(x) \sim 1 \cdot \sin(1 \cdot \pi x) + 1 \cdot \sin(2\pi x)$$

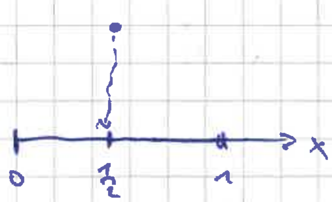
$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-2\pi^2 n^2 t}$$

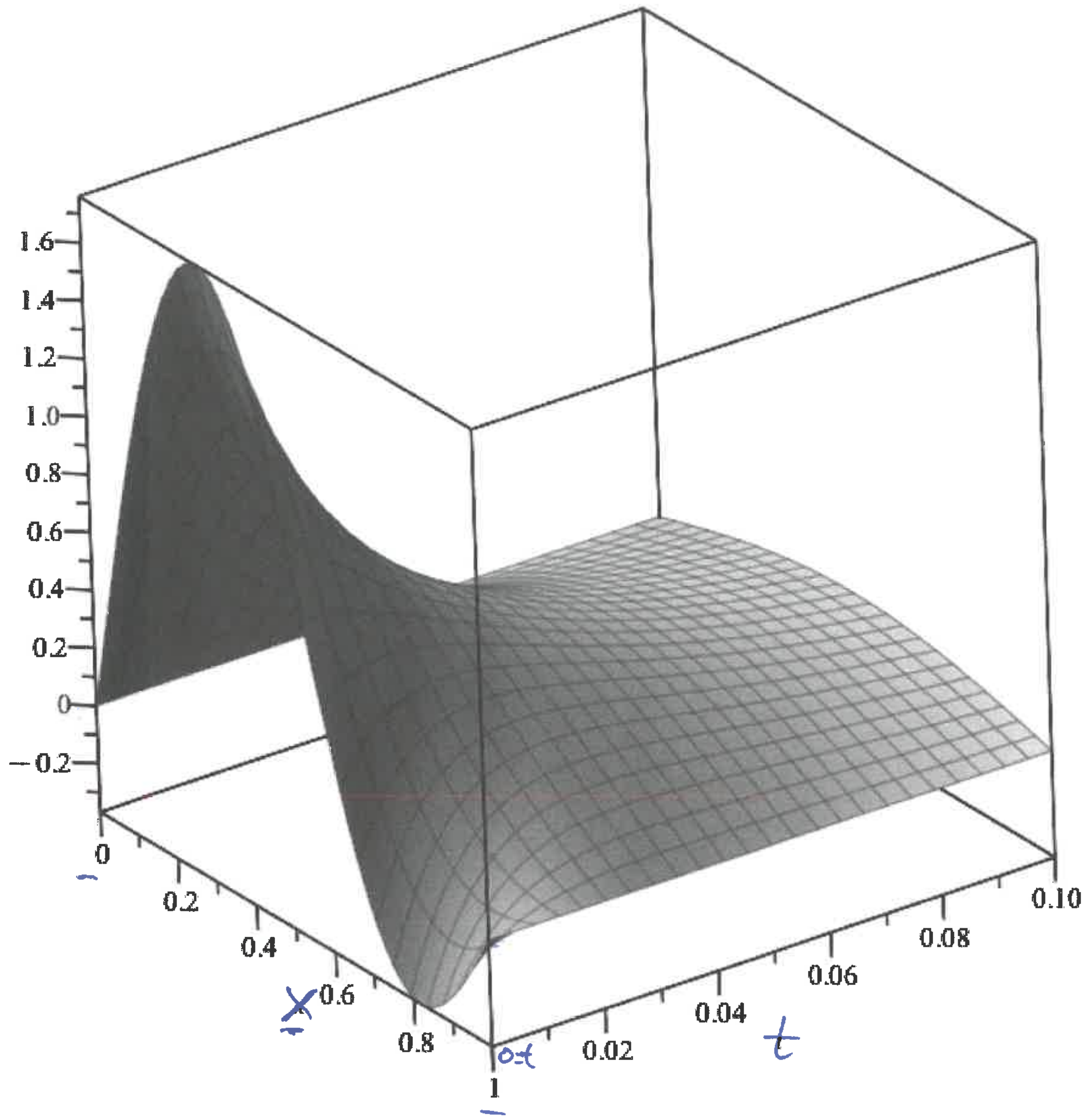
$$= \sin(\pi x) \cdot e^{-2\pi^2 t} + \sin(2\pi x)\, e^{-8\pi^2 t}$$

(b) $$u\left(\tfrac{1}{2}, t\right) = \underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{=1} \cdot e^{-2\pi^2 t} + \underbrace{\sin(\pi)}_{=0}\, e^{-8\pi^2 t}$$

$$= e^{-2\pi^2 t}$$





1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
−0.2
0
0.2
0.4
0.6
0.8
1
x
0.02
0.04
0.06
0.08
0.10
t
0=t

VÜ 6

A 1.6

$$u_t = u_{xx} + \underbrace{1}_{= r(x,t)} \qquad u(0,t) = u(\pi,t) = 0 \qquad t \geq 0$$

$$\underline{u(x,0) = 0} \qquad x \in (0,\pi)$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{a^2 n^2 \pi^2}{L^2} t}$$

$a^2 = 1 \quad L = \pi$

$$= \sum_{n=1}^{\infty} b_n(t) \sin(nx) e^{-n^2 t}$$

$$b_n'(t) = \tilde{b}_n(t) e^{\frac{a^2 n^2 \pi^2 t}{L^2}} = \tilde{b}_n(t) e^{n^2 t}$$

$$\tilde{b}_n(t) = \frac{2}{L} \int_0^L r(x,t) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin(nx)\, dx$$

$$= \frac{2}{\pi} \left[ -\frac{1}{n} \cos(nx) \right]_0^{\pi}$$

$$= \frac{2}{n\pi} \left( -\cos(n\pi) + \cos(0) \right)$$

$$= \frac{2}{n\pi} \left( (-1)^{n+1} + 1 \right) = \begin{cases} 0 & n \text{ gerade} \\ \frac{4}{n\pi} & n \text{ ungerade} \end{cases}$$

$$\Rightarrow \quad b_n'(t) = \frac{2}{n\pi} \left( (-1)^{n+1} + 1 \right) \cdot e^{n^2 t}$$

$$b_n(t) = \frac{2}{n\pi} \left( (-1)^{n+1} + 1 \right) \int e^{n^2 t}\, dt$$

$$= \frac{2}{n\pi} \left( (-1)^{n+1} + 1 \right) \left( \frac{\pi}{n^2} e^{n^2 t} \right) + c_n$$

$$\cancel{= \frac{2}{n^3 \pi} \left( (-1)^{n+1} + 1 \right) \left( e^{n^2 t} + n^2 c_n \right)}$$

$$= \frac{2}{n^3 \pi} \left( (-1)^{n+1} + 1 \right) \cdot e^{n^2 t} + c_n$$

$$u(x,t) = \sum_{n=1}^{\infty} \underline{b_n(t)} \sin(nx)\, e^{-n^2 t}$$

$$= \sum_{n=1}^{\infty} \underbrace{\left[\frac{2}{n^3\pi}\left((-1)^{n+1}+1\right)\cdot e^{n^2 t} + c_n\right]}_{b_n(t)} \sin(nx)\, e^{-n^2 t}$$

homogener Fall:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cdot \sin(nx)\, e^{-n^2 t}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(\xi)\cdot \sin(n\xi)\, d\xi \qquad f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

Wir müssen $c_n$ so wählen, dass $f(x) \sim \sum_{n=1}^{\infty} b_n(0) \sin(nx)$

Fourierreihenentwicklung von $f(x) = 0$:

$$f(x) \sim \sum_{n=1}^{\infty} \underline{0} \cdot \sin(nx)$$

$$\Rightarrow b_n(0) \overset{!}{=} 0$$

$$b_n(0) = \frac{2}{n^3\pi}\left((-1)^{n+1}+1\right)\underbrace{e^{n^2\cdot 0}}_{=1} + c_n$$

$$= \frac{2}{n^3\pi}\left((-1)^{n+1}+1\right) + c_n \overset{!}{=} 0$$

$$\Leftrightarrow \quad c_n = -\frac{2}{n^3\pi}\left((-1)^{n+1}+1\right)$$

Also:

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{n^3\pi}\left((-1)^{n+1}+1\right)\left(e^{n^2 t}-1\right)\cdot \sin(nx)\cdot e^{-n^2 t}\right]$$

$$= \sum_{n=1}^{\infty} \frac{2}{n^3\pi}\left((-1)^{n+1}+1\right)\left(1 - e^{-n^2 t}\right)\cdot \sin(nx)$$

VÜ 6

A17

$u_{tt} = \underbrace{4}_{c^2} u_{xx}$    $u(x,0) = \cos(x)$

(a) Satz von d'Alembert:

$u(x,t) = F(x-ct) + G(x+ct)$    $c=2$

$= F(x-2t) + G(x+2t)$

$u(x,0) = F(x) + G(x) \overset{!}{=} \cos(x)$

z.B.: $F(x) = \cos(x)$ , $G(x) = 0$

$u(x,t) = F(x-2t) + G(x+2t) = \cos(x-2t) + 0$

$= \cos(x-2t)$

$u_t(x,t) = \partial_t \cos(x-2t) = +2 \cdot \sin(x-2t)$

$u_{tt} = -4\cos(x-2t)$

$u_x(x,t) = \partial_x \cos(x-2t) = -\sin(x-2t)$

$u_{xx}(x,t) = -\cos(x-2t)$

$u_{tt} = -4\cos(x-2t) = 4 \cdot \underbrace{(-\cos(x-2t))}_{u_{xx}} = 4u_{xx}$ ✓

$u(x,0) = \cos(x-2\cdot 0) = \cos(x)$ ✓

(b) $F(x) = 0$    $G(x) = \cos(x)$

z.B.: $u(x,t) = \cos(x+2t)$

z.B.: $F(x) = \frac{1}{2}\cos(x)$  $G(x) = \frac{1}{2}\cos(x)$ ; z.B. $F(x) = \cos(x) + e^{x^2}$

$G(x) = -e^{+x^2}$

$u(x,t) = \frac{1}{2}\cos(x-2t) + \frac{1}{2}\cos(x+2t)$ ; $u(x,t) = \cos(x-2t) + e^{(x-2t)^2} - e^{(x+2t)^2}$