HM Vortragsübung 6   17.1.2025

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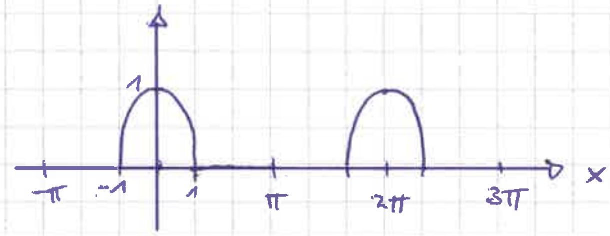
a)



b)

$$a_0 = \frac{2}{\pi}\int_0^1 (1-x^2)\,dx = \frac{2}{\pi}\left[x - \frac{1}{3}x^3\right]_0^1 = \frac{2}{\pi}\left(1-\frac{1}{3}\right) = \frac{4}{3\pi}$$

$k > 0$

$$a_k = \frac{2}{\pi}\int_0^1 (1-x^2)\cos(kx)\,dx = \frac{2}{\pi}\int_0^1 \cos(kx)\,dx - \frac{2}{\pi}\int_0^1 \underset{u}{x^2}\,\underset{v'}{\cos(kx)}\,dx$$

$$= \frac{2}{\pi}\left[\frac{1}{k}\sin(kx)\right]_0^1 - \frac{2}{\pi}\left(\left[x^2\frac{1}{k}\sin(kx)\right]_0^1 - 2\int_0^1 \underset{u}{x}\,\underset{v'}{\frac{1}{k}\sin(kx)}\,dx\right)$$

$$= \frac{2}{k\pi}\sin(k) - \frac{2}{k\pi}\sin(k) + \frac{4}{k\pi}\left(\left[-x\frac{1}{k}\cos(kx)\right]_0^1 + \int_0^1 \frac{1}{k}\cos(kx)\,dx\right)$$

$$= \frac{4}{k\pi}\left(\frac{-1}{k}\cos(k) + \frac{1}{k^2}\left[\sin(kx)\right]_0^1\right)$$

$$= \frac{-4}{\pi k^2}\cos(k) + \frac{4}{\pi k^3}\sin(k) = \frac{4\sin(k) - 4k\cos(k)}{\pi k^3}$$

$$\mathrm{Fourier}_f(x) = \frac{2}{3\pi} + \sum_{k=1}^{\infty} \frac{4\sin(k) - 4k\cos(k)}{\pi k^3}\cos(kx)$$

c)

f stetig bei $x = 0$ ✓ ↗ 7.3.3

$$1 = f(0) = \mathrm{Fourier}_f(0) = \frac{2}{3\pi} + \sum_{k=1}^{\infty}\frac{4\sin(k)-4k\cos(k)}{\pi k^3}\underbrace{\cos(k\cdot 0)}_{=1}$$

$$= \frac{2}{3\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k)-k\cos(k)}{k^3}$$

$$\sum_{k=1}^{\infty}\frac{\sin(k)-k\cos(k)}{k^3} = \pi\,\frac{1-\frac{2}{3\pi}}{4} = \frac{\pi - 2/3}{4} = \frac{1}{4}\pi - 1/6$$

d) ↗ 7.5.4

$$\int_{-1}^{1}(1-x^2)^2\,dx = \int_{-1}^{1} 1 - 2x^2 + x^4\,dx = \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1}$$

$$= 1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5} = 2 - \frac{4}{3} + \frac{2}{5} = \frac{16}{15}$$

$$= \pi\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 = \frac{8}{9}\pi + \cancel{\pi}\sum_{k=1}^{\infty}\frac{16(\sin(k) - k\cos(k))^2}{\pi^2 k^6}$$

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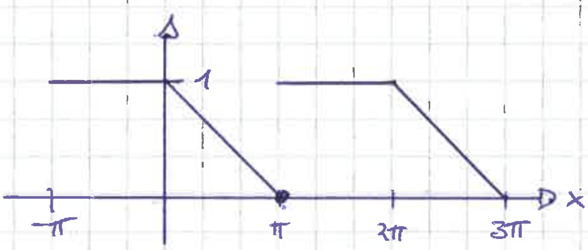
a)



b) $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\pi + \frac{\pi}{2}\right) = \frac{3}{4}$

$k \neq 0$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{0} e^{-ikx}\,dx + \frac{1}{2\pi}\int_0^{\pi}\left(1-\frac{1}{\pi}x\right)e^{-ikx}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{-ik}e^{-ikx}\right]_{-\pi}^{0} + \frac{1}{2\pi}\int_0^{\pi} e^{-ikx}\,dx - \frac{1}{2\pi^2}\int_0^{\pi} \underset{u}{x}\,\underset{v'}{e^{-ikx}}\,dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{-ik}(1-\underbrace{e^{ik\pi}}_{(-1)^k})\right) + \frac{1}{2\pi}\left[\frac{1}{-ik}e^{-ikx}\right]_0^{\pi} - \frac{1}{2\pi^2}\left(\left[\frac{xe^{-ikx}}{-ik}\right]_0^{\pi} - \int_0^{\pi}\frac{1}{-ik}e^{-ikx}\,dx\right)$$

$$= \cancel{\frac{1}{-2ik\pi}(1-(-1)^k)} + \cancel{\frac{1}{-2ik\pi}((-1)^k-1)} - \frac{1}{2\pi^2}\left(\frac{1}{-ik}\pi(-1)^k - \frac{1}{(-ik)^2}\left[e^{-ikx}\right]_0^{\pi}\right)$$

$$= \frac{1}{2ik\pi}(-1)^k + \frac{1}{-2k^2\pi^2}((-1)^k-1)$$

$$= \frac{1-(-1)^k}{2k^2\pi^2} - i\,\frac{(-1)^k}{2k\pi}$$

$$\text{Fourier}_f^{\mathbb{C}}(x) = \frac{3}{4} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left(\frac{1-(-1)^k}{2k^2\pi^2} - i\,\frac{(-1)^k}{2k\pi}\right)e^{ikx}$$

c) $a_0 = 2c_0 = \frac{3}{2}$ $\nearrow$ 7.6.3

$k > 0$

$$a_k = 2\,\mathrm{Re}(c_k) = \frac{1-(-1)^k}{k^2\pi^2}$$

$$b_k = -2\,\mathrm{Im}(c_k) = \frac{(-1)^k}{k\pi}$$

$$\text{Fourier}_f(x) = \frac{3}{4} + \sum_{k=1}^{\infty}\frac{1-(-1)^k}{k^2\pi^2}\cos(kx) + \sum_{k=1}^{\infty}\frac{(-1)^k}{k\pi}\sin(kx)$$

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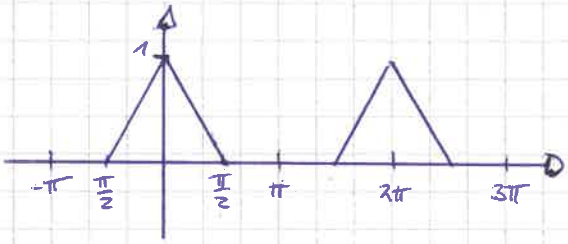
a)



b)

$$a_0 = \frac{2}{\pi}\int_0^{\pi/2}\left(1-\frac{2}{\pi}x\right)dx = \frac{2}{\pi}\frac{\pi}{2} - \frac{4}{\pi^2}\left[\frac{1}{2}x^2\right]_0^{\pi/2}$$

$$= 1 - \frac{2}{\pi^2}\frac{\pi^2}{4} = 1 - \frac{1}{2} = \frac{1}{2}$$

$k>0$

$$a_k = \frac{2}{\pi}\int_0^{\pi/2}\left(1-\frac{2}{\pi}x\right)\cos(kx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\cos(kx)\,dx - \frac{4}{\pi^2}\int_0^{\pi/2}\underset{u}{x}\,\underset{v'}{\cos(kx)}\,dx$$

$$= \frac{2}{\pi}\left[\frac{1}{k}\sin(kx)\right]_0^{\pi/2} - \frac{4}{\pi^2}\left(\left[\frac{1}{k}x\sin(kx)\right]_0^{\pi/2} - \frac{1}{k}\int_0^{\pi/2}\sin(kx)\,dx\right)$$

$$= \cancel{\frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right)} - \cancel{\frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right)} + \frac{4}{k\pi^2}\left[\frac{-1}{k}\cos(kx)\right]_0^{\pi/2}$$

$$= \frac{-4}{k^2\pi^2}\left(\cos\left(\frac{k\pi}{2}\right) - 1\right)$$

$$\text{Fourier}_f(x) = \frac{1}{4} - \sum_{k=1}^{\infty}\frac{4}{k^2\pi^2}\left(\cos\left(\frac{k\pi}{2}\right)-1\right)\cos(kx)$$

c) $\text{Fourier}_g(x) = +\sum_{k=1}^{\infty}\frac{4}{k\pi^2}\left(\cos\left(\frac{k\pi}{2}\right)-1\right)\sin(kx)$ ↗ 7.4.7

d) $\text{Fourier}_g\left(\frac{\pi}{2}\right) = \frac{1}{2}\lim_{x\to\frac{\pi}{2}-0} g(x) + \frac{1}{2}\lim_{x\to\frac{\pi}{2}+0} g(x)$ ↗ 7.3.3

$$= \frac{1}{2}\lim_{x\to\frac{\pi}{2}-0} f'(x) + \frac{1}{2}\lim_{x\to\frac{\pi}{2}+0} f'(x)$$

$$= \frac{-\frac{2}{\pi}+0}{2} = \frac{-1}{\pi}$$

da $\text{Fourier}_g(x) = f'(x)$ wo f diff.bar