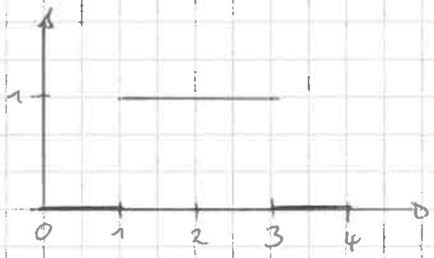
Vortragsübung 7 29.1.2025

19

a)



$L = 4 \qquad T = 8 \qquad \omega = \frac{2\pi}{T} = \frac{\pi}{4}$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin(k\omega x)\,dx = 2\cdot\frac{2}{8}\int_0^4 f(x)\sin\left(k\frac{\pi}{4}x\right)dx \qquad \nearrow 7.7.1$$

$$= \frac{1}{2}\int_1^3 \sin\left(k\frac{\pi}{4}x\right)dx = \frac{1}{2}\left[\frac{-4}{k\pi}\cos\left(k\frac{\pi}{4}x\right)\right]_1^3$$

$$= \frac{-2}{k\pi}\left(\cos\left(\frac{3k\pi}{4}\right) - \cos\left(\frac{k\pi}{4}\right)\right)$$

$$u(x,0) = \sum_{k=1}^{\infty}\frac{2}{k\pi}\left(\cos\left(\frac{k\pi}{4}\right) - \cos\left(\frac{3k\pi}{4}\right)\right)\sin\left(k\frac{\pi}{4}x\right)$$

b) $u_t = 3u_{xx} \qquad a^2 = 3 \qquad \nearrow 8.2.6$

$$u(x,t) = \sum_{k=1}^{\infty}\frac{2}{k\pi}\left(\cos\left(\frac{k\pi}{4}\right) - \cos\left(\frac{3k\pi}{4}\right)\right)\sin\left(k\frac{\pi}{4}x\right)e^{-\frac{3k^2\pi^2}{16}t}$$

c)

$$u_t(x,t) = \sum_{k=1}^{\infty}\frac{-2}{k\pi}\left(\cos\left(\frac{k\pi}{4}\right) - \cos\left(\frac{3k\pi}{4}\right)\right)\sin\left(k\frac{\pi}{4}x\right)\frac{3k^2\pi^2}{16}e^{-\frac{3k^2\pi^2}{16}t}$$

$$= \sum_{k=1}^{\infty}\frac{-3k\pi}{8}\left(\cos\left(\frac{k\pi}{4}\right) - \cos\left(\frac{3k\pi}{4}\right)\right)\sin\left(k\frac{\pi}{4}x\right)e^{-\frac{3k^2\pi^2}{16}t}$$

$$u_x(x,t) = \sum_{k=1}^{\infty}\frac{2}{k\pi}\left(\cos\left(\frac{k\pi}{4}\right) - \cos\left(\frac{3k\pi}{4}\right)\right)\frac{k\pi}{4}\cos\left(k\frac{\pi}{4}x\right)e^{-\frac{3k^2\pi^2}{16}t}$$

$$= \sum_{k=1}^{\infty}\frac{1}{2}\left(\cos\left(\frac{k\pi}{4}\right) - \cos\left(\frac{3k\pi}{4}\right)\right)\cos\left(k\frac{\pi}{4}x\right)e^{-\frac{3k^2\pi^2}{16}t}$$

$$u_{xx}(x,t) = \sum_{k=1}^{\infty}\frac{-1}{2}\left(\cos\left(\frac{k\pi}{4}\right) - \cos\left(\frac{3k\pi}{4}\right)\right)\frac{k\pi}{4}\sin\left(k\frac{\pi}{4}x\right)e^{-\frac{3k^2\pi^2}{16}t}$$

$$= \sum_{k=1}^{\infty}\frac{-k\pi}{8}\left(\cos\left(\frac{k\pi}{4}\right) - \cos\left(\frac{3k\pi}{4}\right)\right)\sin\left(k\frac{\pi}{4}x\right)e^{-\frac{3k^2\pi^2}{16}t}$$

$u_t(x,t) = 3\,u_{xx}(x,t)$ ✓

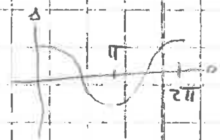
↗ 8.2.10

201

Ansatz: $u(x,t) = \sum_{k=1}^{\infty} b_k(t) \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{a^2k^2\pi^2}{L^2}t}$

$a^2 = 1 \qquad L = \pi \qquad r(x,t) = t$

$\tilde{b}_k = \frac{2}{\pi}\int_0^\pi t \sin(kx)\,dx = \frac{2}{\pi} t \left[\frac{-1}{k}\cos(kx)\right]_0^\pi$



$= \frac{-2}{k\pi} t \left((-1)^k - 1\right) = \frac{2}{k\pi} t \left(1-(-1)^k\right)$

$b_k'(t) = \tilde{b}_k(t)\, e^{\frac{1\cdot k^2\pi^2}{\pi^2}t} = \frac{2}{k\pi} t \left(1-(-1)^k\right) e^{k^2 t}$

$b_k(t) = \frac{2}{k\pi}\left(1-(-1)^k\right)\int \underset{u}{t}\, \underset{v'}{e^{k^2t}}\,dt$

$= \frac{2}{k\pi}\left(1-(-1)^k\right)\left(\left[t\frac{1}{k^2}e^{k^2t}\right] - \frac{1}{k^2}\int e^{k^2t}\,dt\right)$

$= \frac{2}{k\pi}\left(1-(-1)^k\right)\left(\left[t\frac{1}{k^2}e^{k^2t}\right] - \frac{1}{k^4}\left[e^{k^2t}\right]\right)$

$= \frac{2}{k^3\pi}\left(1-(-1)^k\right)\left(t e^{k^2t} - \frac{1}{k^2}e^{k^2t}\right) + c_k \qquad c_k \in \mathbb{R}$

$= \frac{2}{k^3\pi}\left(1-(-1)^k\left(t-\frac{1}{k^2}\right)e^{k^2t} + c_k$

Anfangsbedingung: $u(x,0) = 0 = \sum_{k=1}^{\infty} 0\cdot\sin(kx)$

$\overset{!}{=} \sum_{k=1}^{\infty} b_k(0)\sin(kx)\underbrace{e^{-k^2\cdot 0}}_{=1}$

$0 \overset{!}{=} b_k(0) = \frac{2}{k^3\pi}\left(1-(-1)^k\right)\left(0-\frac{1}{k^2}\right)e^{k^2\cdot 0} + c_k$

$= \frac{2}{k^5\pi}\left((-1)^k-1\right) + c_k$

$\Rightarrow c_k = \frac{2}{k^5\pi}\left(1-(-1)^k\right)$

$u(x,t) = \sum_{k=1}^{\infty}\left(\frac{2}{k^3\pi}\left(1-(-1)^k\right)\left(t-\frac{1}{k^2}\right)e^{k^2t} + \frac{2}{k^5\pi}\left(1-(-1)^k\right)\right)\sin(kx)\,e^{-k^2t}$

$= \sum_{k=1}^{\infty} \frac{2}{k^3\pi}\left(1-(-1)^k\right)\left(\left(t-\frac{1}{k^2}\right) + \frac{1}{k^2}e^{-k^2t}\right)\sin(kx)$

21

$$u_{tt} = 2\,u_{xx} \qquad u(x,0) = \sin(x) \quad \Rightarrow c^2 = 2 \quad \Rightarrow c = \sqrt{2}$$

a) d'Alembert ↗ 8.3.2

$u(x,t) = F(x+ct) + G(x-ct)$

wähle $F(\xi) = \sin(\xi) \qquad G(\eta) = 0$

$u(x,t) = \sin(x+ct) = \sin(x+\sqrt{2}t)$

$u_t(x,t) = \sqrt{2}\cdot\cos(x+\sqrt{2}t) \qquad u_x(x,t) = \cos(x+\sqrt{2}t)$

$u_{tt}(x,t) = -2\,\sin(x+\sqrt{2}t) \qquad u_{xx}(x,t) = -\sin(x+\sqrt{2}t)$

$\Rightarrow u_{tt} = 2\,u_{xx}$ ✓

$u(x,0) = \sin(x+\sqrt{2}\cdot 0) = \sin(x)$ ✓

b) • $F(\xi) = 0 \qquad G(\xi) = \sin(\xi) \quad \Rightarrow u(x,t) = \sin(x-\sqrt{2}t)$

$u_t(x,t) = -\sqrt{2}\cos(x-\sqrt{2}t) \qquad u_x(x,t) = \cos(x-\sqrt{2}t)$

$u_{tt}(x,t) = -2\sin(x-\sqrt{2}t) \qquad u_{xx}(x,t) = -\sin(x-\sqrt{2}t)$

$\Rightarrow u_{tt} = 2\,u_{xx}$ ✓

$u(x,0) = \sin(x-\sqrt{2}\cdot 0) = \sin(x)$ ✓

• $F(\xi) = \frac{1}{2}\sin(\xi) \qquad G(\eta) = \frac{1}{2}\sin(\eta) \quad \Rightarrow u(x,t) = \frac{1}{2}\sin(x+\sqrt{2}t) + \frac{1}{2}\sin(x-\sqrt{2}t)$

$u_t(x,t) = \frac{\sqrt{2}}{2}\cos(x+\sqrt{2}t) - \frac{\sqrt{2}}{2}\cos(x-\sqrt{2}t)$

$u_{tt}(x,t) = -\sin(x+\sqrt{2}t) - \sin(x-\sqrt{2}t)$

$u_x(x,t) = \frac{1}{2}\cos(x+\sqrt{2}t) + \frac{1}{2}\cos(x-\sqrt{2}t)$

$u_{xx}(x,t) = -\frac{1}{2}\sin(x+\sqrt{2}t) - \frac{1}{2}\sin(x-\sqrt{2}t)$

$\Rightarrow u_{tt} = 2\,u_{xx}$ ✓

$u(x,0) = \frac{1}{2}\sin(x+\sqrt{2}\cdot 0) + \frac{1}{2}\sin(x-\sqrt{2}\cdot 0) = \sin(x)$ ✓