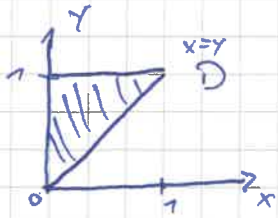
VÜ Blatt 1

A1



$D = \{(x,y) \in [0,1]^2 : x \le y\}$

Normalbereich bzgl x-Achse:

$$D = \{(x,y) \in \mathbb{R}^2 : a \le x \le b \ , \ g(x) \le y \le h(x)\}$$

Bei uns: $\underset{a}{0} \le x \le \underset{b}{1}$ $\qquad$ $\underset{g(x)}{x} \le y \le \underset{h(x)}{1}$

Normalbereich bzgl. der y-Achse:

$$D = \{(x,y) \in \mathbb{R}^2 : a \le y \le b \ , \ g(y) \le x \le h(y)\}$$

$\underset{a}{0} \le y \le \underset{b}{1}$, $\qquad$ $\underset{g(y)}{0} \le x \le \underset{h(y)}{y}$

$f: [0,1]^2 \to \mathbb{R}$ mit $f(x,y) = \sqrt{xy}$

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \int_x^1 \sqrt{xy}\,dy\,dx$$

$$= \int_0^1 \sqrt{x} \left(\int_x^1 \sqrt{y}\,dy \right) dx = \int_0^1 \sqrt{x} \left[\frac{2}{3} y^{\frac{3}{2}} \right]_x^1 dx$$

$$= \int_0^1 \sqrt{x} \cdot \left(\frac{2}{3} - \frac{2}{3} x^{\frac{3}{2}} \right) dx$$

$$= \frac{2}{3} \int_0^1 \sqrt{x} - \sqrt{x} \cdot x^{\frac{3}{2}}\,dx = \frac{2}{3} \int_0^1 \sqrt{x} - x^{\frac{1}{2}+\frac{3}{2}}\,dx$$

$$= \frac{2}{3} \int_0^1 \sqrt{x} - x^2\,dx = \frac{2}{3} \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right]_0^1$$

$$= \frac{2}{3} \left(\frac{2}{3} - \frac{1}{3} - 0 \right) = \frac{2}{3} \cdot \frac{1}{3} = \underline{\underline{\frac{2}{9}}}$$

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \int_0^y \sqrt{xy}\,dx\,dy$$

$$= \int_0^1 \left(\int_0^y \sqrt{y} \cdot \sqrt{x} \right) dx\,dy = \int_0^1 \left[\sqrt{y} \cdot \frac{2}{3} x^{\frac{3}{2}} \right]_0^y dy$$

$$= \int_0^1 \sqrt{y} \cdot \frac{2}{3} y^{\frac{3}{2}} - 0\,dy = \frac{2}{3} \int_0^1 y^2\,dy = \frac{2}{3} \left[\frac{1}{3} y^3 \right]_0^1$$

$$= \frac{2}{3} \cdot \left(\frac{1}{3} - 0 \right) = \underline{\underline{\frac{2}{9}}}$$

FALSCH:

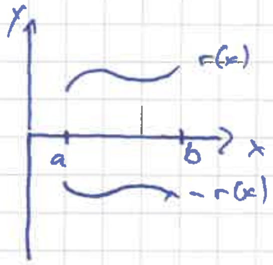
$$\int_x^1 \int_0^1 \sqrt{xy}\,dx\,dy = \int_x^1 \sqrt{y} \left(\int_0^1 \sqrt{x}\,dx \right) dy$$

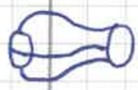
$$= \int_{\cancel{x}}^1 \sqrt{y} \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^1 dy = \int_x^1 \sqrt{\cancel{x}y} \cdot \frac{2}{3}\,dy$$

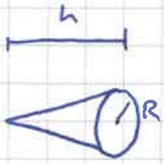
$$= \frac{2}{3} \left[\frac{2}{3} y^{\frac{3}{2}} \right]_x^1 = \frac{2}{3} \left(\frac{2}{3} - \frac{2}{3} x^{\frac{3}{2}} \right)$$

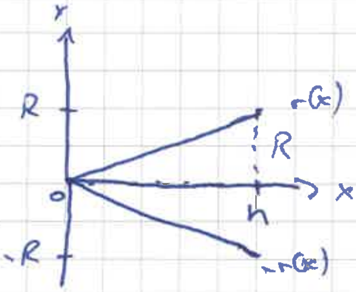
A2

(a) $V = \pi \int_a^b r(x)^2 \, dx$









$a = 0 \, , \, b = h$

$r(x) = mx + c \overset{c=0}{=} mx = \frac{R}{h} x$

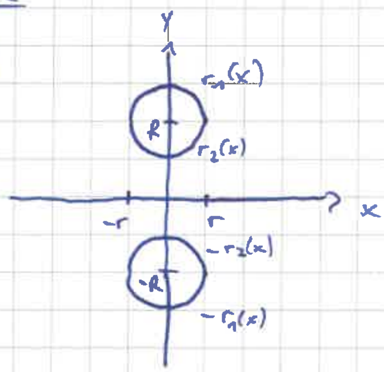
$V = \pi \int_a^b r(x)^2 \, dx = \pi \int_0^h \left(\frac{R}{h} x\right)^2 dx$

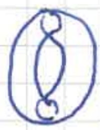
$= \pi \int_0^h \frac{R^2}{h^2} x^2 \, dx = \frac{R^2}{h^2} \pi \int_0^h x^2 dx = \frac{R^2}{h^2} \pi \left[\frac{1}{3} x^3\right]_0^h$

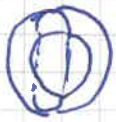
$= \frac{R^2}{h^2} \pi \left(\frac{1}{3} h^3 - 0\right) = \frac{\pi}{3} \cdot R^2 h$

A2

(b)







$$V = \pi \int_a^b r(x)^2 \, dx = \pi \int_{-r}^{r} r_1(x)^2 \, dx - \pi \int_{-r}^{r} r_2(x)^2 \, dx$$

$$= \pi \int_a^b r_1(x)^2 - r_2(x)^2 \, dx = \pi \int_a^b \big(r_1(x) + r_2(x)\big)\big(r_1(x) - r_2(x)\big) \, dx$$

$$= \pi \int_{-r}^{r} 2R \cdot 2\sqrt{r^2 - x^2} \, dx$$

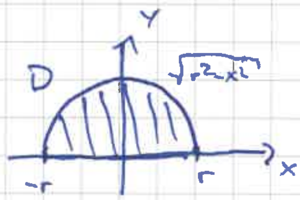
$$= 4R\pi \int_{-r}^{r} \underbrace{\sqrt{r^2 - x^2}} \, dx = 4\pi R \overbrace{\int_{-r}^{r} \int_0^{\sqrt{r^2-x^2}} 1 \, dy \, dx}^{= \frac{1}{2}\pi r^2}$$

$$\int_0^{\sqrt{r^2-x^2}} 1 \, dy = \big[y \big]_0^{\sqrt{r^2-x^2}} = \sqrt{r^2-x^2} - 0$$

NR

$$\int_{-r}^{r} \int_0^{\sqrt{r^2-x^2}} 1 \, dy \, dx = \iint_D 1 \, dx\,dy = V(D)$$

$$-r \le x \le r, \quad 0 \le y \le \sqrt{r^2 - x^2}$$



$$V(D) = \frac{1}{2}\pi r^2$$

$$= 4\pi R \cdot \frac{1}{2}\pi r^2 = 2\pi^2 R r^2$$

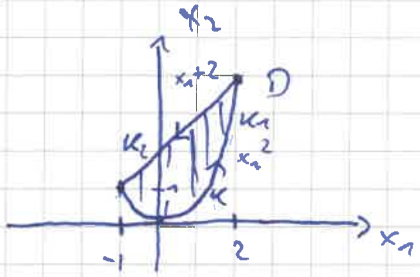
übung:

$$\int_{-r}^{r} \sqrt{r^2 - x^2} \, dx \quad \text{mit Substitution} \quad x = r \cdot \sin(\varphi)$$

$$\int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2 \sin^2\varphi} \cdot r\cos(\varphi) \, d\varphi$$

43

(a)



$-1 \le x_1 \le 2$ $\quad$ $x_1^2 \le x_2 \le x_1+2$

(b) Parametrisierung von K_1:

$C_1 : [-1,2] \to \mathbb{R}^2, t \mapsto \begin{pmatrix} t \\ t^2 \end{pmatrix}$

Parametrisierung von K_2

$C_2 : [-1,2] \to \mathbb{R}^2, t \mapsto \begin{pmatrix} t \\ t+2 \end{pmatrix}$ nicht richtig orientiert

Richtige Orientierung: $t \mapsto -t$ und Intervalle $[a,b] \mapsto [-b,-a]$

$\tilde{C}_2 : [-2,+1] \to \mathbb{R}^2, \quad t \mapsto \begin{pmatrix} -t \\ -t+2 \end{pmatrix}$

$C_2 : [2,-1]$

Angenommen wir wollen die ~~Q.~~ Orientierung von C_1 ändern:

$\tilde{C}_1 : [-2,+1] \to \mathbb{R}^2, \quad t \mapsto \begin{pmatrix} -t \\ (-t)^2 \end{pmatrix} = \begin{pmatrix} -t \\ t^2 \end{pmatrix}$

A3

(c) $z(g,K) = \int_K g(x) \cdot dx = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} g(c(t)) \cdot c'(t)\, dt$

Bei uns:

$z(g,K) = \int_{K_1} g(x) \cdot dx + \int_{K_2} g(x) \cdot dx$

$= \int_{-1}^{2} \begin{pmatrix} t - t^2 \\ t^2 \cdot t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt$

$+ \int_{-2}^{1} \begin{pmatrix} -t - (-t+2) \\ (-t)^2 \cdot (-t+2) \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} dt$

$= \int_{-1}^{2} (t - t^2) \cdot 1 + (t^4) \cdot (2t)\, dt$

$+ \ldots$

$g(x_1, x_2) = \begin{pmatrix} x_1 - x_2 \\ x_1^2 x_2 \end{pmatrix}$

$c_1(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$

$c_1'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$

$\tilde{c}_2(t) = \begin{pmatrix} -t \\ -t+2 \end{pmatrix}$

$\tilde{c}_2'(t) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

(c)

$$I(g,K) = \int_K g(x)\cdot dx = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} g(c(t))\cdot c'(t)\,dt$$

in unserem Fall:

$$I(g,K) = \int_{K_1} g(x)\cdot dx + \int_{K_2} g(x)\cdot dx$$

$$= \int_{-1}^{2} g(C_1(t))\cdot C_1'(t)\,dt + \int_{-2}^{1} g(C_2(t))\cdot C_2'(t)\,dt$$

$$C_1'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} ; \quad C_2'(t) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$C_1(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} ; \quad C_2(t) = \begin{pmatrix} -t \\ -t+2 \end{pmatrix}$$

Also:

$$I(g,K) = \int_{-1}^{2} \begin{pmatrix} t-t^2 \\ t^4 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2t \end{pmatrix} dt + \int_{-2}^{1} \begin{pmatrix} -t-(-t+2) \\ (-t)^2(-t+2) \end{pmatrix}\cdot\begin{pmatrix} -1 \\ -1 \end{pmatrix} dt$$

$$= \int_{-1}^{2} t-t^2+2t^5\,dt + \int_{-2}^{1} \begin{pmatrix} -2 \\ -t^3+2t^2 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ -1 \end{pmatrix} dt$$

$$= \left[\frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{3}t^6\right]_{-1}^{2} + \int_{-2}^{1} 2+t^3-2t^2\,dt$$

$$= 2 - \frac{8}{3} + \frac{64}{3} - \frac{1}{2} - \left(+\frac{1}{3}\right) - \frac{1}{3} + \left[2t + \frac{1}{4}t^4 - \frac{2}{3}t^3\right]_{-2}^{1}$$

$$= 2 + \frac{56}{3} - \frac{1}{2} - \frac{2}{3} + 2 + \frac{1}{4} - \frac{2}{3} - (-4) - 4 - \left(+\frac{\cancel{86}}{3}\right)$$

$$= 4 - \frac{1}{2} + \frac{1}{4} + \frac{38}{3} = \cancel{\frac{24}{6}\ \frac{3}{6}}$$

$$= \cancel{\frac{48}{12} - \frac{6}{12} + \frac{3}{12} + \frac{152}{12} - \frac{187}{12}}$$

$$= 16 - \frac{2}{4} + \frac{1}{4} = \frac{63}{4}$$

Aufgabe 3 $\quad x,y = x_1, x_2 \quad$ Achtung: Aufgabe mit $x_1 x_2$ gestellt, nicht x,y

(d) $\operatorname{rot} g : D \to \mathbb{R} : (x,y) \mapsto \operatorname{rot} g(x,y) = \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}$

Bei uns: $\quad g(x,y) = \begin{pmatrix} x-y \\ x^2 y \end{pmatrix}$

$$\frac{\partial g_2}{\partial x} = 2xy \quad ; \quad \frac{\partial g_1}{\partial y} = -1$$

$$\operatorname{rot} g(x,y) = 2xy + 1$$

$$\iint_D \operatorname{rot} g \, dx\, dy = \int_{-1}^{2} \int_{x^2}^{x+2} 2xy + 1 \, dy \, dx$$

$$= \int_{-1}^{2} \left[xy^2 + y \right]_{x^2}^{x+2} dx = \int_{-1}^{2} x(x+2)^2 + x + 2 - x^5 - x^2 \, dx$$

$$= \int_{-1}^{2} x^3 + 4x^2 + 4x + x + 2 - x^5 - x^2 \, dx$$

$$= \int_{-1}^{2} -x^5 + x^3 + 3x^2 + 5x + 2 \, dx$$

$$= \left[-\frac{1}{6}x^6 + \frac{1}{4}x^4 + x^3 + \frac{5}{2}x^2 + 2x \right]_{-1}^{2}$$

$$= -\frac{64}{6} + 4 + 8 + 10 + 4 - \left(-\frac{1}{6}\right) - \left(\frac{1}{4}\right) - (-1) - \frac{5}{2} - (-2)$$

$$= -\frac{63}{6} + 29 - \frac{1}{4} - \frac{5}{2} = -\frac{42}{4} + \frac{116}{4} - \frac{1}{4} - \frac{10}{4} = \frac{63}{4}$$

(e) Nach dem Satz von Green gilt

$$Z(g,K) = \int_K g(x) \cdot dx = \iint_D \operatorname{rot} g \, dx dy$$

~~Dass~~ Unsere Ergebnisse aus (c) und (d) sollten also übereinstimmen und das tun sie auch.