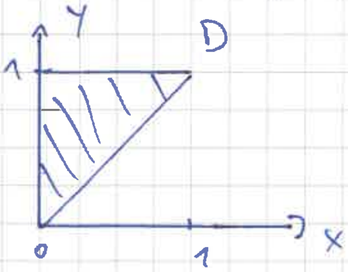
VÜ Blatt 1



$$D = \{ (x,y) \in [0,1]^2 : x \le y \}$$

Normalbereich bzgl der x-Achse

$$D = \{ (x,y) \in \mathbb{R}^2 : a \le x \le b , g(x) \le y \le h(x) \}$$

Bei uns:

$$D = \{ (x,y) \in \mathbb{R}^2 : \underset{\substack{\downarrow \\ a}}{0} \le x \le \underset{\substack{\downarrow \\ b}}{1} , \underset{\substack{\downarrow \\ g(x)}}{x} \le y \le \underset{\substack{\downarrow \\ h(x)}}{1} \}$$

Normalbereich bzgl der y-Achse

$$D = \{ (x,y) \in \mathbb{R}^2 : a \le y \le b , g(y) \le x \le h(y) \}$$

Bei uns:

$$D = \{ (x,y) \in \mathbb{R}^2 : \underset{\substack{\downarrow \\ a}}{0} \le y \le \underset{\substack{\downarrow \\ b}}{1} , \underset{\substack{\downarrow \\ g(y)}}{0} \le x \le \underset{\substack{\downarrow \\ h(y)}}{y} \}$$

$f: [0,1]^2 \to \mathbb{R}$, $(x,y) \longmapsto f(x,y) = \sqrt{xy}$

Zu berechnen: $\iint\limits_D f(x,y)\, dx\, dy$

$$\iint\limits_D f(x,y)\,dx\,dy = \int_0^1 \int_x^1 \sqrt{xy}\; dy\,dx$$

$$= \int_0^1 \int_x^1 \sqrt{x} \cdot \sqrt{y}\; dy\,dx = \int_0^1 \sqrt{x} \cdot \left(\int_x^1 \sqrt{y}\, dy \right) dx$$

$$= \int_0^1 \sqrt{x} \cdot \left[\frac{2}{3} y^{\frac{3}{2}} \right]_x^1 dx = \int_0^1 \sqrt{x} \left(\frac{2}{3} - \frac{2}{3} x^{\frac{3}{2}} \right) dx$$

$$= \int_0^1 \frac{2}{3}\sqrt{x} - \frac{2}{3}\sqrt{x} \cdot x^{\frac{3}{2}}\, dx = \int_0^1 \frac{2}{3}\sqrt{x} - \frac{2}{3} x^{\frac{1}{2}+\frac{3}{2}}\, dx$$

$$= \int_0^1 \frac{2}{3}\sqrt{x} - \frac{2}{3} x^2\, dx = \frac{2}{3}\left[\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right]_0^1$$

$$= \frac{2}{3}\left[\frac{2}{3} - \frac{1}{3} - 0 \right] = \frac{2}{3} \cdot \frac{1}{3} = \underline{\underline{\frac{2}{9}}}$$

$$\sqrt{x} \cdot \sqrt{x^3} = \sqrt{x \cdot x^3} = \sqrt{x^4} = x^2$$

$$\iint\limits_D f(x,y)\,dx\,dy = \int_0^1 \int_0^y \sqrt{xy}\; dx\,dy$$

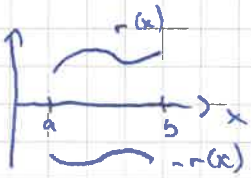
$$= \int_0^1 \left(\int_0^y \sqrt{y} \cdot \sqrt{x}\, dx \right) dy = \int_0^1 \left[\sqrt{y} \cdot \frac{2}{3} x^{\frac{3}{2}} \right]_0^y dy$$

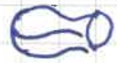
$$= \int_0^1 \Big(\frac{2}{3}\sqrt{y} \cdot y^{\frac{3}{2}} - \underbrace{\frac{2}{3}\sqrt{y} \cdot 0}_{=0} \Big)\, dy = \int_0^1 \frac{2}{3} y^2\, dy$$

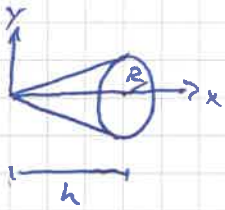
$$= \frac{2}{3}\left[\frac{1}{3} y^3 \right]_0^1 = \frac{2}{3} \cdot \left(\frac{1}{3} - 0 \right) = \frac{2}{9}$$

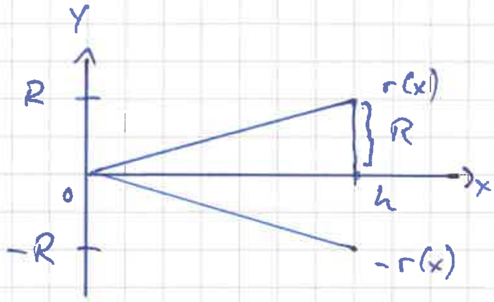
Aufgabe 2

(a) $$V = \pi \int_a^b r(x)^2 \, dx$$



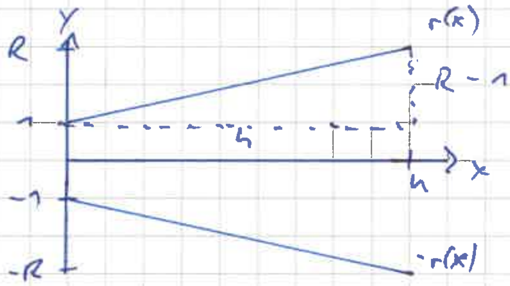


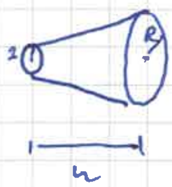




$$0 \le x \le h \; ; \quad r(x) = mx + c \overset{c=0}{=} mx = \frac{R}{h} x$$

Nebenbsp.:





$$0 \le x \le h \; ; \quad r(x) = mx + c = mx + 1 = \frac{R-1}{h} x + 1$$

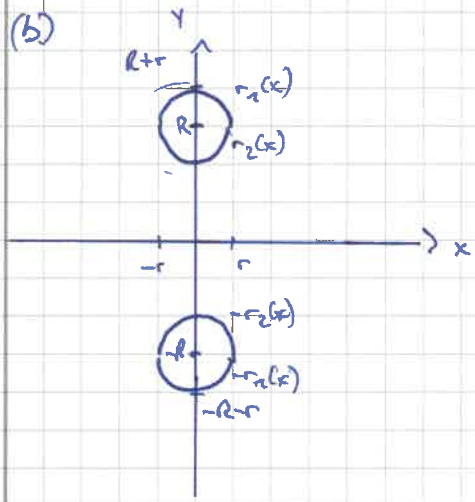
$$V = \pi \cdot \int_a^b r(x)^2 \, dx = \pi \int_0^h \left(\frac{R}{h} \cdot x \right)^2 dx$$

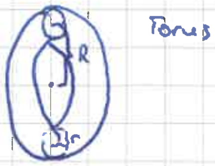
$$= \pi \int_0^h \frac{R^2}{h^2} \cdot x^2 \, dx = \pi \frac{R^2}{h^2} \int_0^h x^2 \, dx = \pi \frac{R^2}{h^2} \left[\frac{1}{3} x^3 \right]_0^h$$

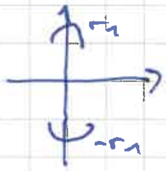
$$= \pi \frac{R^2}{h^2} \left(\frac{1}{3} h^3 - 0 \right) = \frac{\pi}{3} R^2 \cdot h$$

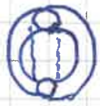
Aufgabe 2

(b)









$$V = \pi \int_a^b r(x)^2 \, dx$$

$$= \pi \int_a^b r_1(x)^2 \, dx - \pi \int_a^b r_2(x)^2 \, dx$$

$$= \pi \int_a^b r_1(x)^2 - r_2(x)^2 \, dx = \pi \int_a^b \underbrace{(r_1(x) + r_2(x))}_{2R} \cdot \underbrace{(r_1(x) - r_2(x))}_{2\sqrt{r^2-x^2}} \, dx$$

$$= \pi \int_a^b 2R \cdot 2\sqrt{r^2-x^2} \, dx$$

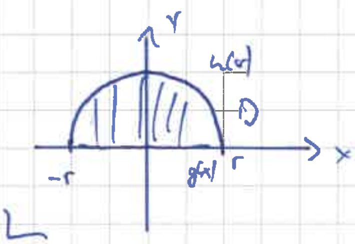
$$= \pi \int_{-r}^{r} 4R\sqrt{r^2-x^2} \, dx = 4R\pi \int_{-r}^{r} \sqrt{r^2-x^2} \, dx$$

$$= \cancel{4\pi R \int_{-r}^{r} \int_0^{\sqrt{r^2-x^2}}} \quad 4\pi R \int_{-r}^{r} \int_0^{\sqrt{r^2-x^2}} 1 \, dy \, dx$$

NR:

$$\int_{-r}^{r} \int_0^{\sqrt{r^2-x^2}} 1 \, dy \, dx = \int_{-r}^{r} [y]_0^{\sqrt{r^2-x^2}} \, dx = \underline{\int_{-r}^{r} \sqrt{r^2-x^2} - 0 \, dx}$$

$$D = \{(x,y) \in \mathbb{R}^2 : \underset{a}{-r} \le x \le \underset{b}{r}, \ \underset{g(x)}{0} \le y \le \underset{h(x)}{\sqrt{r^2-x^2}}\}$$



$$V(D) = \frac{1}{2}\pi r^2$$

$$= 4\pi R \cdot V(D) = 4\pi R \cdot \left(\frac{1}{2}\pi r^2\right) = 2\pi^2 R r^2$$

Alternative: Substitution

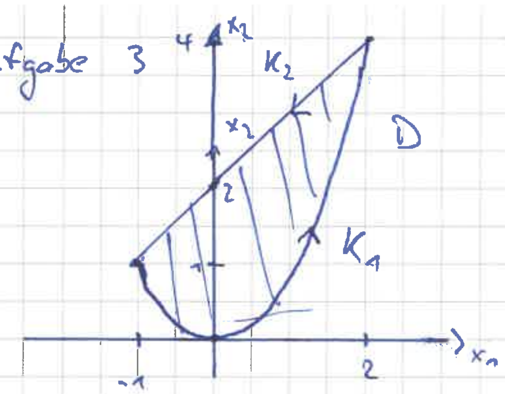
$x = r \cdot \sin\varphi$

$\frac{dx}{d\varphi} = r\cos\varphi \Leftrightarrow dx = r\cos\varphi \, d\varphi$

$$\int_{-r}^{r} \sqrt{r^2-x^2} \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2(\sin\varphi)^2} \cdot r\cos\varphi \, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\sqrt{1-\sin^2\varphi} \cdot r\cos\varphi \, d\varphi$$

Aufgabe 3

(a)



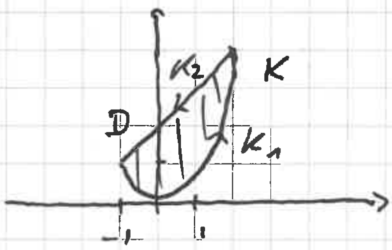
$y = f(x) = x^2$

Parametrisierung von K_1:

$c_1 : [-1, 2] \to \mathbb{R}^2 , \quad t \mapsto \begin{pmatrix} t \\ t^2 \end{pmatrix}$

Aufgabe 3

(a)



(b) Zerlege $K = K_1 \cup K_2$

Parametrisierung von K_1:

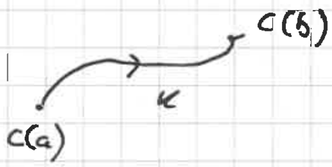
$c_1: [-1,2] \to \mathbb{R}^2$ mit $c_1(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$

Parametrisierung von K_2:

Erste Idee: falsch herum: $c_2: [-1,2] \to \mathbb{R}^2$ mit $c_2(t) = \begin{pmatrix} t \\ t+2 \end{pmatrix}$

$t \to -t$ $\quad c_2: [-2,1] \to \mathbb{R}^2$ mit $c_2(t) = \begin{pmatrix} -t \\ -t+2 \end{pmatrix}$

Allgemein:



$c: [a,b] \to \mathbb{R}^2$

$\tilde{c}: [-b,-a] \to \mathbb{R}^2$ $\quad \tilde{c}(t) = c(-t)$

Dann $\tilde{c}(-b) = c(b)$, $\tilde{c}(-a) = c(a)$

$\cancel{\tilde{c}(t)}$, $t \in [-b,-a]$: $\tilde{c}(t) = c(-t)$ mit $-t \in [a,b]$

(c)

$$Z(g,K) = \int_K g(x) \cdot dx = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} g(c(t)) \cdot c'(t)\, dt$$

in unserem Fall:

$$Z(g,K) = \int_{K_1} g(x) \cdot dx + \int_{K_2} g(x) \cdot dx$$

$$= \int_{-1}^{2} g(c_1(t)) \cdot c_1'(t)\, dt + \int_{-2}^{1} g(c_2(t)) \cdot c_2'(t)\, dt$$

$$c_1'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \;;\; c_2'(t) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$c_1(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} \;;\; c_2(t) = \begin{pmatrix} -t \\ -t+2 \end{pmatrix}$$

Also:

$$Z(g,K) = \int_{-1}^{2} \begin{pmatrix} t - t^2 \\ t^4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt + \int_{-2}^{1} \begin{pmatrix} -t-(-t+2) \\ (-t)^2(-t+2) \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} dt$$

$$= \int_{-1}^{2} t - t^2 + 2t^5\, dt + \int_{-2}^{1} \begin{pmatrix} -2 \\ -t^3 + 2t^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} dt$$

$$= \left[\frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{3}t^6\right]_{-1}^{2} + \int_{-2}^{1} 2 + t^3 - 2t^2\, dt$$

$$= 2 - \frac{8}{3} + \frac{64}{3} - \frac{1}{2} - \left(+\frac{1}{3}\right) - \frac{1}{3} + \left[2t + \frac{1}{4}t^4 - \frac{2}{3}t^3\right]_{-2}^{1}$$

$$= 2 + \frac{56}{3} - \frac{1}{2} - \frac{2}{3} + 2 + \frac{1}{4} - \frac{2}{3} - (-4) - 4 - \left(+\frac{16}{3}\right)$$

$$= 4 - \frac{1}{2} + \frac{1}{4} + \frac{38}{3} =$$

$$= \cancel{\frac{48}{12} - \frac{6}{12} + \frac{3}{12} + \frac{152}{12} = \frac{197}{12}}$$

$$= 16 - \frac{2}{4} + \frac{1}{4} = \frac{63}{4}$$

Aufgabe 3 $\quad x,y = x_1, x_2$ $\quad$ Achtung: Aufgabe mit $x_1 x_2$ gestellt, nicht x, y

(d) $\operatorname{rot} g : D \to \mathbb{R} : (x,y) \mapsto \operatorname{rot} g(x,y) = \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}$

Bei uns: $\quad g(x,y) = \begin{pmatrix} -x-y \\ x^2 y \end{pmatrix}$

$\frac{\partial g_2}{\partial x} = 2xy \; ; \; \frac{\partial g_1}{\partial y} = -1$

$\operatorname{rot} g(x,y) = 2xy + 1$

$$\iint_D \operatorname{rot} g \, dx\, dy = \int_{-1}^{2} \int_{x^2}^{x+2} 2xy + 1 \, dy\, dx$$

$$= \int_{-1}^{2} \left[x y^2 + y \right]_{x^2}^{x+2} dx = \int_{-1}^{2} x(x+2)^2 + x + 2 - x^5 - x^2 \, dx$$

$$= \int_{-1}^{2} x^3 + 4x^2 + 4x + x + 2 - x^5 - x^2 \, dx$$

$$= \int_{-1}^{2} -x^5 + x^3 + 3x^2 + 5x + 2 \, dx$$

$$= \int_{-1}^{2} \left[-\frac{1}{6}x^6 + \frac{1}{4}x^4 + x^3 + \frac{5}{2}x^2 + 2x \right]_{-1}^{2}$$

$$= -\frac{64}{6} + 4 + 8 + 10 + 4 - \left(-\frac{1}{6}\right) - \left(\frac{1}{4}\right) - (-1) - \frac{5}{2} - (-2)$$

$$= -\frac{63}{6} + 29 - \frac{1}{4} - \frac{5}{2} = -\frac{42}{4} + \frac{116}{4} - \frac{1}{4} - \frac{10}{4} = \frac{63}{4}$$

(e) Nach dem Satz von Green gilt

$$\tau(g, K) = \int_K g(x) \cdot dx = \iint_D \operatorname{rot} g \, dx dy$$

~~Dass~~ Unsere Ergebnisse aus (c) und (d) sollten also übereinstimmen und das tun sie auch.