A9

(a) $\mathcal{L}(u(t)) = U(s) = \frac{1}{p(s)} = \frac{1}{s^3 - s^2 + 5s - 5}$

(b) $\mathcal{L}\left(e^{at}\cos(\omega t)\right) = \frac{s-a}{(s-a)^2+\omega^2}$

$\mathcal{L}(e^{at}) = \frac{1}{s-a}$

$\mathcal{L}\left(e^{at}\sin(\omega t)\right) = \frac{\omega}{(s-a)^2+\omega^2}$

NS von $p(s)$: $\lambda_1 = 1$

Polynomdivision: $(s^3 - s^2 + 5s - 5) : (s-1) = s^2 + 5$

$$\begin{array}{l} s^3 - s^2 \\ \hline \quad 0 \\ \qquad 5s - 5 \\ \qquad 5s - 5 \\ \hline \qquad\quad 0 \end{array}$$

Also: $s^3 - s^2 + 5s - 5 = (s-1)\underline{(s^2+5)}$

PBZ: $\frac{1}{(s-1)(s^2+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+5}$

$= \frac{A(s^2+5)}{(s-1)(s^2+5)} + \frac{(Bs+C)\cdot(s-1)}{(s-1)(s^2+5)}$

$= \frac{(A+B)s^2 + (-B+C)s + (5A-C)}{(s-1)(s^2+5)}$

Koeffizientenvergleich:

$$\left.\begin{array}{l} A+B \overset{!}{=} 0 \\ -B+C \overset{!}{=} 0 \\ 5A-C \overset{!}{=} 1 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} A = -B \\ B = C \\ A = -C \\ 6A = 1 \end{array}\right\} \Leftrightarrow \begin{array}{l} C = -\frac{1}{6} \\ B = -\frac{1}{6} \\ A = \frac{1}{6} \end{array}$$

Also: $\frac{1}{(s-1)(s^2+5)} = \frac{1}{6}\cdot\frac{1}{s-1} + \left(-\frac{1}{6}\right)\cdot\frac{s+1}{s^2+5}$

$\underbrace{-\frac{1}{6}\,\frac{s}{s^2+5}}_{a=0 \quad \omega^2=5} \qquad \underbrace{-\frac{1}{6\omega}\,\frac{+1\cdot\omega}{s^2+5}}_{a=0 \quad \omega^2=5}$

$\mathcal{L}^{-1}\left(\frac{1}{(s-1)(s^2+5)}\right) = \frac{1}{6}\,e^{1\cdot t}$
$- \frac{1}{6}\,e^{0\cdot t}\cos(\sqrt{5}\cdot t)$
$+ \left(-\frac{1}{6\sqrt{5}}\,e^{0\cdot t}\sin(\sqrt{5}\,t)\right)$

$\mathcal{L}\left(e^{at}\cos(\omega t)\right) = \frac{s-a}{(s-a)^2+\omega^2}$

$\mathcal{L}\left(e^{at}\sin(\omega t)\right) = \frac{\omega}{(s-a)^2+\omega^2}$

Also:

(b) $u(t) = \frac{1}{6} e^{t} - \frac{1}{6} \cos(\sqrt{5}\, t) - \frac{1}{6\sqrt{5}} \sin(\sqrt{5}\, t)$

(c) Aus Skript: $f(t) = (u * g)(t)$ $\qquad g(t) = 3$

$$(u*g)(t) = (g*u)(t) = \int_0^t g(t-\tau)\cdot u(\tau)\, d\tau$$

$$= \int_0^t 3\cdot\left(\frac{1}{6} e^{\tau} - \frac{1}{6}\cos(\sqrt{5}\tau) - \frac{1}{6\sqrt{5}}\cdot \sin(\sqrt{5}\tau)\right) d\tau$$

$$= \int_0^t \frac{1}{2} e^{\tau}\, d\tau + \int_0^t -\frac{1}{2}\cos(\sqrt{5}\tau)\, d\tau \cancel{+\int_0^t d\tau}$$

$$+ \int_0^t -\frac{1}{2\sqrt{5}} \sin(\sqrt{5}\tau)\, d\tau$$

$$= \left[\frac{1}{2} e^{\tau}\right]_0^t + \left[-\frac{1}{2}\frac{1}{\sqrt{5}} \sin(\sqrt{5}\tau)\right]_0^t$$

$$+ \left[-\frac{1}{2\sqrt{5}}\left(-\frac{1}{\sqrt{5}} \cos(\sqrt{5}\tau)\right)\right]_0^t$$

$$= \frac{1}{2} e^{t} - \frac{1}{2} + \left(-\frac{1}{2\sqrt{5}}\right)\sin(\sqrt{5}\, t) - 0$$

$$+ \left(\frac{1}{2\cdot 5}\cos(\sqrt{5}\, t)\right) - \frac{1}{2\cdot 5}$$

$$= \frac{1}{2} e^{t} - \frac{1}{2\sqrt{5}} \sin(\sqrt{5}\, t) + \frac{1}{10}\cos(\sqrt{5}\, t) \underbrace{- \frac{1}{2} - \frac{1}{10}}_{= -\frac{6}{10} = -\frac{3}{4}}$$

$$= \frac{1}{2} e^{t} - \frac{1}{2\sqrt{5}} \sin(\sqrt{5}\, t) + \frac{1}{10}\cos(\sqrt{5}\, t) - \frac{3}{4}$$

A1①

$$y' - \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}_{A} y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a)

EW von A: $\lambda_1 = 2$, $\lambda_2 = 2$

EV von A:

$$(A - \lambda \cdot E_2) \cdot v_2 \overset{!}{=} 0$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Da $rg\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1$ gibt es nur EV von der

Form $s \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $s \in \mathbb{R}$

(b) $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$B^0 \cdot v = v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B^1 \cdot v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$B^2 \cdot v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(c) Aus VL Fundamentalsystem: $B^{k-1} v \neq 0$, $B^k v = 0$ $k=2$

$$f_{[1]}(x) = e^{\lambda x} \left(\frac{x^0}{0!} B^{k-1} v \right) = e^{2x} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f_{[2]}(x) = e^{\lambda x} \left(\frac{x^1}{1!} B^{k-1} v + \frac{x^0}{0!} B^{k-2} v \right) = e^{2x} \left(x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Allgemeine homogene Lsg:

$$y_h(x) = e^{2x} \begin{pmatrix} c_1 + c_2 x \\ c_2 \end{pmatrix}$$

(d) Variation der Konstanten

$$W_{sys}(x) \cdot \begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = b(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$W_{sys}(x) = \begin{pmatrix} f_{[1]} & f_{[2]} \end{pmatrix} = \begin{pmatrix} e^{2x} & xe^{2x} \\ 0 & e^{2x} \end{pmatrix}$$

$$f_{[1]}(x) = e^{2x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad f_{[2]}(x) = e^{2x}\begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = W_{sys}^{-1}(x) \cdot b(x)$$

$$W_{sys}^{-1}(x) = \frac{1}{e^{4x}} \cdot \begin{pmatrix} e^{2x} & -xe^{2x} \\ 0 & e^{2x} \end{pmatrix}$$

$$= e^{-2x} \cdot \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$$

Für 3x3 oder höher

$$W_{sys}^{-1}(x) = W^{-1}(0) \cdot W(-x) \cdot W^{-1}(0)$$

$$W_{sys}^{-1}(x) \cdot b(x)$$

$$W_{sys}^{-1}(x) \cdot b(x) = e^{-2x} \cdot \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= e^{-2x}\begin{pmatrix} -x \\ 1 \end{pmatrix} = \begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix}$$

$$c_1(x) = \int -xe^{-2x}\,dx \overset{p.I.}{=} \dots = \left[\frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x}\right]$$

$$c_2(x) = \int e^{-2x}\,dx = \left[-\frac{1}{2}e^{-2x}\right]$$

$$y_p(x) = c_1(x) \cdot f_{[1]}(x) + c_2(x) \cdot f_{[2]}(x) = e^{2x}\begin{pmatrix} c_1(x) + c_2(x) \cdot x \\ c_2(x) \end{pmatrix}$$

$$= \cancel{e^{2x}}\begin{pmatrix} \underline{\frac{1}{2}xe^{-2x}} + \frac{1}{4}\cancel{e^{-2x}} + \underline{\left(-\frac{1}{2}e^{-2x}\right) \cdot x} \\ -\frac{1}{2}\cancel{e^{-2x}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{2} \end{pmatrix}$$

Prüfe: $y_p'(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in DGL: $\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{2} \end{pmatrix} = -\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(e) Allgemeine Lösung des inhomogenen Problems:

$$y(x) = y_p(x) + y_h(x) = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{2} \end{pmatrix} + e^{2x}\begin{pmatrix} c_1 + c_2 \cdot x \\ c_2 \end{pmatrix}$$

$$y(0) = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Leftrightarrow \begin{array}{l} c_1 = \frac{3}{4} \\ c_2 = \frac{3}{2} \end{array}$$

Also

$$y(x) = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{2} \end{pmatrix} + e^{2x}\begin{pmatrix} \frac{3}{4} + \frac{3}{2}x \\ \frac{3}{2} \end{pmatrix}$$

A11

$$y' = \underbrace{\begin{pmatrix} 3 & -1 & 3 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{A} y$$

EW von A:

$$\det(A - \lambda E_3) = \det\begin{pmatrix} (3-\lambda) & -1 & 3 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$= (3-\lambda) \cdot \det\left(\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}\right) = (3-\lambda)\cdot(\lambda^2 + 1)$$

$$\Rightarrow \lambda_1 = 3 \qquad \lambda_{2,3} = \pm i$$

Eigenvektor zu $\lambda_1 = 3$: $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Eigenvektor zu $\lambda_2 = i$:

$$\begin{pmatrix} 3-i & -1 & 3 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{pmatrix} \cdot v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-i & -1 & 3 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{pmatrix} \xrightarrow{2\cdot i} \begin{pmatrix} 3-i & -1 & 3 \\ 0 & -i & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{2\cdot i} \begin{pmatrix} 3-i & 0 & 3-i \\ 0 & -i & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \cdot \frac{1}{3-i} \\ \cdot (+i) \end{array}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_2 = t_* \cdot \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix}$$

$$\stackrel{t=1}{=} \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Fundamentalsystem:

$$f_{[1]}(x) = e^{\lambda_1 x} \cdot v_1 = e^{3x} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$f_{[2]}(x) = e^{\alpha x} \left(\cos(\beta x) \cdot u - \sin(\beta x) \cdot v \right)$$

wobei $\lambda_2 = \underbrace{\alpha}_{=0} + i \underbrace{\beta}_{=1}$ und $w_2 = \underbrace{u}_{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} + i \underbrace{v}_{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

Also: $f_{[2]}(x) = \cos(x) \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \sin(x) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$f_{[3]}(x) = e^{\alpha x} \left(\sin(\beta x) u + \cos(\beta x) v \right)$$

$$= \sin(x) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \cos(x) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Allgemeine Lsg.:

$$y(x) = y_h(x) = c_1 f_{[1]}(x) + c_2 f_{[2]}(x) + c_3 \cdot f_{[3]}(x)$$

$$= \begin{pmatrix} c_1 e^{3x} - c_2 \cos(x) - c_3 \sin(x) \\ -c_2 \sin(x) + c_3 \cos(x) \\ c_2 \cos(x) + c_3 \sin(x) \end{pmatrix}$$

$\cos(\pi) = -1$
$\sin(\pi) = 0$

$$y(\pi) = \begin{pmatrix} c_1 e^{3\pi} + c_2 \\ -c_3 \\ -c_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Leftrightarrow \begin{matrix} c_2 = c_3 = -1 \\ c_1 e^{3\pi} - 1 = 0 \end{matrix}$$

$$\Leftrightarrow \begin{matrix} c_2 = c_3 = -1 \\ c_1 = \frac{1}{e^{3\pi}} \end{matrix}$$

Also insgesamt:

$$y(x) = \begin{pmatrix} e^{-3\pi} \cdot e^{3x} + \cos(x) + \sin(x) \\ \sin(x) - \cos(x) \\ -\cos(x) - \sin(x) \end{pmatrix}$$