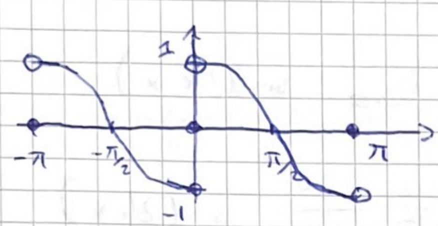
A12

(a) $f(x) = \cos(x) \qquad 0 < x < \pi$



(b) f ungerade $\Rightarrow a_0 = a_j = 0 \qquad j \in \mathbb{N}$

Allg. Fourierreihe:

$$\text{Fourier}_p(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(jx) + \sum_{j=1}^{\infty} b_j \sin(jx)$$

$$f \text{ ungerade} \Rightarrow b_j = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin(jx)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos(x) \cdot \sin(jx)\, dx$$

NR

$$\int_0^{\pi} \underset{v'}{\cos(x)}\, \underset{u}{\sin(jx)}\, dx = \underbrace{\left[\sin(x) \sin(jx)\right]_0^{\pi}}_{=0} - \int_0^{\pi} \sin(x) \cdot j \cos(jx)\, dx$$

$$= -\int_0^{\pi} j \cdot \underset{v'}{\sin(x)}\, \underset{u}{\cos(jx)}\, dx = \left[+j \cos(x) \cos(jx)\right]_0^{\pi}$$

$$+ \int_0^{\pi} (-\cos(x))\,(-j^2 \sin(jx))\, dx$$

$$= \left(-j \cos(j\pi) - j \underbrace{\cos(0)}_{=1}\right) + j^2 \int_0^{\pi} \cos(x) \sin(jx)\, dx$$

$$\Leftrightarrow (1 - j^2) \int_0^{\pi} \cos(x) \sin(jx)\, dx = -j\left(\cos(j\pi) + 1\right)$$

$$\Leftrightarrow \int_0^{\pi} \cos(x) \sin(jx)\, dx = \frac{j}{j^2 - 1} \left(\cos(j\pi) + 1\right) = \begin{cases} 0, & j \text{ ungerade} \\ \frac{2j}{j^2-1}, & j \text{ gerade} \end{cases}$$

$$\Rightarrow b_j = \begin{cases} 0 & j \text{ ungerade} \\ \frac{4}{\pi} \frac{j}{j^2-1} & j \text{ gerade} \end{cases}$$

Also

$$\text{Fourier}_f(x) = \sum_{k=1}^{\infty} b_{2k} \cdot \sin(2k x)$$

$$= \sum_{k=1}^{\infty} \frac{4}{\pi} \frac{2k}{(2k)^2-1} \sin(2kx)$$

$$= \sum_{k=1}^{\infty} \frac{8}{\pi} \frac{k}{4k^2-1} \sin(2kx)$$

(c) $f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$\text{Fourier}_f\left(\frac{\pi}{4}\right) = \sum_{k=1}^{\infty} \frac{8}{\pi} \frac{k}{4k^2-1} \sin\left(2k \cdot \frac{\pi}{4}\right)$$

$$= \sum_{k=1}^{\infty} \frac{8}{\pi} \frac{k}{4k^2-1} \sin\left(k \cdot \frac{\pi}{2}\right)$$

NR

$$\sin\left(k\frac{\pi}{2}\right) = \begin{cases} 0 & k \text{ gerade} \\ (-1)^{j+1} & k = 2j-1 \end{cases}$$

$$= \sum_{j=1}^{\infty} \frac{8}{\pi} \frac{(2j-1)}{4(2j-1)^2-1} \cdot (-1)^{j+1}$$

- f ist bis auf endlich viele Ausnahmestellen $(-\pi, 0, \pi)$ in $(-\pi, \pi)$ stetig diffbar.
- An jeder Ausnahmestelle $x_0 \in \{\pi, 0, -\pi\}$ existieren die einseitigen Grenzwerte $\lim_{x \to x_0+0} f(x)$, $\lim_{x \to x_0-0} f(x)$

$\lim_{x \to x_0+0} f'(x)$, $\lim_{x \to x_0-0} f'(x)$

Bei uns: in allen Punkten ist der links oder rechtsseitige GW ± 1 (Ableitung ist $-\sin(x)$) also sind die Grenzwerte 0

Damit konvergiert f an jeder Stelle x, an der f stetig ist, gegen $f(x)$.

Also

$$f\left(\tfrac{\pi}{4}\right) = \text{Fourier}_f\left(\tfrac{\pi}{4}\right)$$

$$\frac{\sqrt{2}}{2} = \sum_{j=1}^{\infty} \frac{8}{\pi} \cdot (-1)^{j+1} \frac{2j-1}{4(2j-1)^2-1}$$

$$\Leftrightarrow \sum_{j=1}^{\infty} (-1)^{j+1} \frac{2j-1}{4(2j-1)^2-1} = \frac{\sqrt{2}\,\pi}{16}$$

(d) Parsevalsche Gleichung:

$$\|f\|^2 = \int_{-\pi}^{\pi} f(x)^2\,dx = \pi \frac{a_0^2}{2} + \pi \sum_{j=1}^{\infty} (a_j^2 + b_j^2)$$

Bei uns:

$$\|f\|^2 = \int_{-\pi}^{\pi} \cos^2(x)\,dx = \int_{-\pi}^{\pi} \cos(x)\cdot\cos(x)\,dx$$

$$\overset{\text{Orthogonalitätsrelation}}{=} \pi \quad \cancel{\text{Ecke}}$$

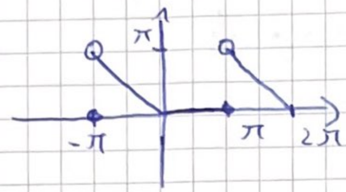
Also:

$$\pi = \pi \cdot \sum_{j=1}^{\infty} \left(\frac{8}{\pi} \frac{j}{4j^2-1}\right)^2$$

$$\Leftrightarrow \frac{\pi^2}{\cancel{16}\,64} = \sum_{j=1}^{\infty} \frac{j^2}{(4j^2-1)^2}$$

A13

(a) $f(x) = \begin{cases} -x & , \quad -\pi < x < 0 \\ 0 & , \quad 0 \leq x \leq \pi \end{cases}$



(b) $\text{Fourier}_f^{\mathbb{C}}(x) = \sum_{k=-\infty}^{\infty} c_k \, e^{ikx}$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{0} \underset{u}{(-x)} \cdot \underset{v'}{e^{-ikx}}\, dx$$

$$\overset{k \neq 0}{=} \frac{1}{2\pi} \underbrace{\left[\frac{-x}{-ik} e^{-ikx} \right]_{-\pi}^{0}}_{= \frac{\pi}{ik} e^{ik\pi}} - \frac{1}{2\pi} \int_{-\pi}^{0} \frac{1}{ik} e^{-ikx}\, dx$$

$$= \frac{1}{2ik} e^{ik\pi} - \frac{1}{2\pi} \left[\frac{1}{k^2} e^{-ikx} \right]_{-\pi}^{0}$$

$$= \frac{1}{2ik} e^{ik\pi} - \frac{1}{2\pi k^2} + \frac{1}{2\pi k^2} e^{ik\pi}$$

$$= -\frac{1}{2\pi k^2} + \underbrace{e^{ik\pi}}_{= \begin{cases} 1 & k \text{ gerade} \\ -1 & k \text{ ungerade} \end{cases}} \left(\frac{1}{2ik} + \frac{1}{2\pi k^2} \right)$$

$$= \begin{cases} \frac{1}{2ik}, & k \text{ gerade} \\ -\frac{1}{2ik} - \frac{1}{\pi k^2}, & k \text{ ungerade} \end{cases}$$

$k=0$: $c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{0} -x \, dx$

$$= \frac{1}{2\pi} \left[-\frac{1}{2} x^2 \right]_{-\pi}^{0} = \frac{\pi}{4}$$

Also

$$\text{Fourier}^{\mathbb{C}}_f(x) = \sum_{k=-\infty}^{\infty} \cancel{\frac{1}{4ik}}\, c_k\, e^{ikx}$$

mit $c_k = \begin{cases} \frac{1}{2ik}, & k \neq 0 \text{ gerade} \\ -\frac{1}{2ik} - \frac{1}{\pi k^2}, & k \text{ ungerade} \\ \frac{\pi}{4}, & k = 0 \end{cases}$

(c)

reelle Fourier-Reihe:

$$a_0 = 2c_0 = \frac{\pi}{2}$$

$$a_k = c_k + c_{-k} = \begin{cases} \frac{1}{2ik} - \frac{1}{2ik} = 0, & k \neq 0 \text{ gerade} \\ -\frac{2}{\pi k^2}, & k \text{ ungerade} \end{cases}$$

$$b_k = i(c_k - c_{-k}) = \begin{cases} i \cdot \left(\frac{2}{2ik}\right) = \frac{1}{k}, & k \text{ gerade} \\ -\frac{1}{k}, & k \text{ ungerade} \end{cases}$$

$$\text{Fourier}_f(x) = \frac{\pi}{4} + \sum_{j=1}^{\infty} -\frac{2}{\pi(2j-1)^2} \cos((2j-1)x)$$

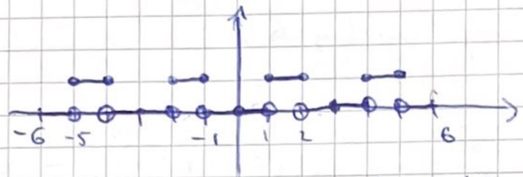
$$+ \sum_{j=1}^{\infty} (-1)^j \cancel{\frac{1}{j}} \frac{1}{j} \sin(jx)$$

A 14

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x \le 2 \\ 0, & 2 < x \le 3 \end{cases}$$

f ist gerade und 6-periodisch

(a)



$T = 6$

gerade $\Rightarrow b_j = 0$

$$\omega = \frac{2\pi}{T} = \frac{\pi}{3}$$

$$a_j = 2 \cdot \frac{2}{6} \int_0^3 f(x) \cos\left(j \frac{\pi}{3} x\right) dx$$

$$= \frac{2}{3} \left(\cancel{\int_{-2}^{-1} \cos\left(j \frac{\pi}{3} x\right) dx} + \int_1^2 \cos\left(j \frac{\pi}{3} x\right) dx \right)$$

$k \neq 0$

$$= \frac{2}{3} \left(\left[\frac{3}{j\pi} \sin\left(j \frac{\pi}{3} x\right) \right]_1^2 \right)$$

$$= \frac{2}{j\pi} \underbrace{\left(\sin\left(\frac{2\pi}{3} j\right) - \sin\left(\frac{\pi}{3} j\right) \right)}_{=: A_j}$$

$$= \cancel{\begin{cases} \frac{2}{j\pi} \sqrt{3} & j \bmod 6 = 1 \text{ oder } j \bmod 6 = 5 \\ 0 & \text{sonst} \end{cases}}$$

$\sin\left(\frac{\pi}{3}\right) =$

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\sin(\pi) = 0$

$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\sin(2\pi) = 0$

$$a_0 = 2 \cdot \frac{1}{3} \int_0^3 f(x)\, dx = \frac{2}{3}$$

NR:

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\sin(\pi) = 0$

$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\sin(2\pi) = 0$

Also

$A_1 = 0$

$A_2 = -\sqrt{3}$

$A_3 = 0$

$A_4 = \sqrt{3}$

$A_5 = 0$

$A_6 = 0$

$$\Rightarrow \quad a_j = \begin{cases} \frac{2}{j\pi} \cdot (-\sqrt{3}), & j \bmod 6 = 2 \\ \frac{2}{j\pi} \cdot (\sqrt{3}), & j \bmod 6 = 4 \\ 0, & \text{sonst} \end{cases} \qquad j \geq 1$$

$$\text{Fourier}_f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(j \omega x)$$

$$= \frac{1}{3} + \sum_{j=1}^{\infty} -\frac{\sqrt{3} \cdot 2}{(6j-4)\pi} \cos\left(\underset{(6j-4)}{j} \frac{\pi}{3} x\right) + \frac{\sqrt{3} \cdot 2}{(6j-2)\pi} \cos\left((6j-2) \frac{\pi}{3} x\right)$$