Bsp In $\mathbb{F}_4 = \{0, 1, \alpha, 1+\alpha\}$

wird:

- $$\alpha^5 = \underbrace{\alpha\cdot\alpha}_{=\alpha+1}\cdot\underbrace{\alpha\cdot\alpha}_{=\alpha+1}\cdot\alpha$$

$$= (\alpha+1)\cdot(\alpha+1)\cdot\alpha$$

$$= (\underbrace{\alpha^2}_{=\alpha+1} + \underbrace{2\alpha}_{=0} + 1)\cdot\alpha$$

$$= (\alpha+\underbrace{2}_{=0})\cdot\alpha$$

$$= \alpha^2 = \alpha+1$$

laut Skript:

- $$\frac{\alpha}{\alpha+1}+1 = \alpha\cdot\underbrace{\frac{1}{\alpha+1}}_{=\alpha}+1$$

$$= \underbrace{\alpha^2}_{=\alpha+1}+1$$

$$= \alpha+\underbrace{2}_{=0} = \alpha$$

Bsp Für $\mathbb{F}_8 = \{0, 1, \beta, 1+\beta, \beta^2, 1+\beta^2, \beta+\beta^2, 1+\beta+\beta^2\}$

wollen wir die Multiplikationstafel aufstellen:

$(\cdot)$	0	1	β	$1+\beta$	β^2	$1+\beta^2$	$\beta+\beta^2$	$1+\beta+\beta^2$
0	0	0	0	0	0	0	0	0
1	0	1	β	$1+\beta$	β^2	$1+\beta^2$	$\beta+\beta^2$	$1+\beta+\beta^2$
β	0	β	β^2	$\beta+\beta^2$	$1+\beta$	1	$1+\beta+\beta^2$	$1+\beta^2$
$1+\beta$	0	$1+\beta$	$\beta+\beta^2$	$1+\beta^2$	$1+\beta+\beta^2$	β^2	1	β
β^2	0	β^2	$1+\beta$	$1+\beta+\beta^2$	$\beta+\beta^2$	β	$1+\beta^2$	1
$1+\beta^2$	0	$1+\beta^2$	1	β^2	β	$1+\beta+\beta^2$	$1+\beta$	$\beta+\beta^2$
$\beta+\beta^2$	0	$\beta+\beta^2$	$1+\beta+\beta^2$	1	$1+\beta^2$	$1+\beta$	β	β^2
$1+\beta+\beta^2$	0	$1+\beta+\beta^2$	$1+\beta^2$	β	1	$\beta+\beta^2$	β^2	$1+\beta$

z.B.: $(1+\beta^2)\cdot(\beta+\beta^2)$

$$= \beta + \beta^2 + \underbrace{\beta^3}_{\beta+1} + \underbrace{\beta^4}_{\beta^2+\beta} = 1 + 3\beta + 2\beta^2 = 1+\beta$$

Bsp In $\mathbb{F}_8$ sei

$$\frac{\beta^5+1}{\beta^2+1} = ?$$ zu berechnen

Jedenfalls ist: $\beta^5+1 = \beta^2\cdot\beta^3+1$

$$= \beta^2(\beta+1)+1 = \beta^3+\beta^2+1$$

$$= (\beta+1)+\beta^2+1 = \beta^2+\beta+2 = \beta^2+\beta$$

Dann wird, laut Multiplikationstafel:

$$(\beta^2+1)\cdot\beta = 1.$$

Also ist $$\frac{\beta^5+1}{\beta^2+1} = \frac{\beta^2+\beta}{\beta^2+1} = (\beta^2+\beta)\cdot\beta$$

$$= \beta^3+\beta^2 = (\beta+1)+\beta^2$$

$$= \beta^2+\beta+1$$

Alternativ kann man auch die Tabelle der Potenzen von β verwenden:

β^0	β^1	β^2	β^3	β^4	β^5	β^6	β^7
1	β	β^2	$\beta+1$	$\beta^2+\beta$	$\beta^2+\beta+1$	β^2+1	1

Dann wird

$$\frac{\beta^5+1}{\beta^2+1} \overset{\text{s.o.}}{=} \frac{\beta^2+\beta}{\beta^2+1} \underset{\substack{\text{der}\\ \text{Potenzen}}}{\overset{\text{Tabelle}}{=}} \frac{\beta^4}{\beta^6}$$

$$= \beta^{-2} = \beta^7 \cdot \beta^{-2} = \beta^5 \underset{\substack{\text{der}\\ \text{Potenzen}}}{\overset{\text{Tabelle}}{=}} \beta^2+\beta+1$$

Bsp In $\mathbb{F}_8$ wird

$$\beta^{54} = \beta^{49+5} = (\beta^7)^7 \cdot \beta^5$$

$$= \beta^5 \underset{\substack{\text{der}\\ \text{Potenzen}}}{\overset{\text{Tabelle}}{=}} \beta^2+\beta+1$$

Bsp für

$$\mathbb{F}_9 = \{0, 1, \underbrace{-1}_{=2}, i, 1+i, -1+i, \underbrace{-i}_{=2i}, 1-i, -1-i\}$$

wollen wir die Multiplikationstafel bestimmen.

$(\cdot)$	0	1	-1	i	$1+i$	$-1+i$	$-i$	$1-i$	$-1-i$
0	0	0	0	0	0	0	0	0	0
1	0	1	-1	i	$1+i$	$-1+i$	$-i$	$1-i$	$-1-i$
-1	0	-1	1	$-i$	$-1-i$	$1-i$	i	$-1+i$	$1+i$
i	0	i	$-i$	-1	$-1+i$	$-1-i$	1	$1+i$	$1-i$
$1+i$	0	$1+i$	$-1-i$	$-1+i$	$-i$	1	$1-i$	-1	i
$-1+i$	0	$-1+i$	$1-i$	$-1-i$	1	i	$1+i$	$-i$	-1
$-i$	0	$-i$	i	1	$1-i$	$1+i$	-1	$-1-i$	$-1+i$
$1-i$	0	$1-i$	$-1+i$	$1+i$	-1	$-i$	$-1-i$	i	1
$-1-i$	0	$-1-i$	$1+i$	$1-i$	i	-1	$-1+i$	1	$-i$

z.B.: $(-1+i)(-1+i) = (-1)^2 + 2(-1)i + i^2$

$= 1 - 2i - 1 = i$

Bsp

Für $a \in \mathbb{F}_3$ ist:

a	0	1	-1
a^3	0	1	-1

Also ist $a^3 = a$ für $a \in \mathbb{F}_3$.

Damit wird in $\mathbb{F}_9$, mit $a, b \in \mathbb{F}_3$:

$$(a+bi)^3 = a^3 + \underbrace{3}_{0}a^2bi + \underbrace{3}_{0}a(bi)^2 + (bi)^3$$

$$= a^3 + b^3 i^3$$

$$= a + b(-i)$$

$$= a - bi$$