<u>Bsp</u>

Sei $A := \begin{pmatrix} 0 & i & i & i & i \\ -i & 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 & i \\ -i & -i & -i & -i & 0 \end{pmatrix} \in \mathbb{C}^{5\times 5}$.

Es ist A hermitesch: $\overline{A}^t = A$.

Wir wollen A unitär diagonalisieren.

$$\chi_A(X) = \det \begin{array}{cc} \begin{pmatrix} -X & i & i & i & i \\ -i & -X & 0 & 0 & i \\ -i & 0 & -X & 0 & i \\ -i & 0 & 0 & -X & i \\ -i & -i & -i & -i & -X \end{pmatrix} & \begin{array}{l} +1 \\ \\ (-1) \\ (-1) \\ \\ \end{array} \end{array}$$

$$= \det \begin{pmatrix} -X-i & 0 & 0 & 0 & -X+i \\ -i & -X & 0 & 0 & i \\ 0 & X & -X & 0 & 0 \\ 0 & X & 0 & -X & 0 \\ -i & -i & -i & -i & -X \end{pmatrix}$$

$$= \ldots$$

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$$\ldots = X^2 \det \begin{pmatrix} -X-i & 0 & 0 & 0 & -X+i \\ -i & -X & 0 & 0 & i \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -i & -i & -i & -i & -X \end{pmatrix} \leftarrow (-i)\ (-i)$$

$$= X^2 \det \begin{pmatrix} -X-i & 0 & 0 & 0 & -X+i \\ -i & -X & 0 & 0 & i \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -i & -3i & 0 & 0 & -X \end{pmatrix}$$

$$= X^2 \det \begin{pmatrix} -X-i & 0 & -X+i \\ -i & -X & i \\ -i & -3i & -X \end{pmatrix} \leftarrow -1$$

$$= X^2 \det \begin{pmatrix} -X & X & -X \\ -i & -X & i \\ -i & -3i & -X \end{pmatrix}$$

$= \ldots$

$$\ldots = X^3 \det \begin{pmatrix} -1 & 1 & -1 \\ -i & -X & i \\ -i & -3i & -X \end{pmatrix}$$

$$= X^3 \det \begin{pmatrix} -1 & 0 & 0 \\ -i & -X-i & 2i \\ -i & -4i & -X+i \end{pmatrix}$$

$$= X^3 \cdot \det(-1) \cdot \det \begin{pmatrix} -X-i & 2i \\ -4i & -X+i \end{pmatrix}$$

$$= X^3 \cdot (-1) \cdot \left(X^2 + 1 - 8 \right)$$

$$= - X^3 \left(X^2 - 7 \right) = -X^3 (X - \sqrt{7})(X + \sqrt{7})$$

$\Rightarrow$ Eigenwerte

$\lambda_1 = 0$ mit $aV_A(0) = 3$

$\vdots$

⋮

$\lambda_2 = \sqrt{7}$ mit $aV_A(\sqrt{7}) = 1$

$\lambda_3 = -\sqrt{7}$ mit $aV_A(-\sqrt{7}) = 1$

- Orthonormalbasis in $E_A(0)$:

$$\left(\begin{array}{ccccc|c} 0 & i & i & i & i & 0 \\ -i & 0 & 0 & 0 & i & 0 \\ -i & 0 & 0 & 0 & i & 0 \\ -i & 0 & 0 & 0 & i & 0 \\ -i & -i & -i & -i & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\Longrightarrow \quad E_A(0) = {}_{\mathbb{C}}\left\langle \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

Nun Gram-Schmidt in seiner hermiteschen Version.

Da aber alle Vektoreinträge zufällig reell sind, ist das hier dasselbe wie Gram-Schmidt in der reellen Version:

$$d_1' = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad d_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$d_2' = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \underbrace{\left(\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}^t \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right) \frac{1}{\sqrt{2}}}_{-1/2} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} \Rightarrow d_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$d_3' = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \underbrace{\left(\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}^t \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) \frac{1}{\sqrt{2}}}_{-1/2} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$- \underbrace{\left(\frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 2 \\ 0 \end{pmatrix}^t \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) \frac{1}{\sqrt{6}}}_{-1/6} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix} \Rightarrow \quad d_3 = \frac{1}{\sqrt{21}} \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \\ 3 \end{pmatrix}$$

$\Rightarrow$ $E_4(0)$ hat Orthonormalbasis

$$(d_1, d_2, d_3) = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{21}} \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \\ 3 \end{pmatrix} \right)$$

• Orthonormalbasis in $E_A(\sqrt{7})$:

$$
\begin{array}{c}
\left(\begin{array}{ccccc|c}
-\sqrt{7} & i & i & i & i & 0 \\
-i & -\sqrt{7} & 0 & 0 & i & 0 \\
-i & 0 & -\sqrt{7} & 0 & i & 0 \\
-i & 0 & 0 & -\sqrt{7} & i & 0 \\
-i & -i & -i & -i & -\sqrt{7} & 0
\end{array}\right)
\end{array}
\begin{array}{c}
\\ \\ -1 \\ -1 \\ \\
\end{array}
$$

$$
\rightsquigarrow \left(\begin{array}{ccccc|c}
-\sqrt{7} & i & i & i & i & 0 \\
-i & -\sqrt{7} & 0 & 0 & i & 0 \\
0 & \sqrt{7} & -\sqrt{7} & 0 & 0 & 0 \\
0 & \sqrt{7} & 0 & -\sqrt{7} & 0 & 0 \\
-i & -i & -i & -i & -\sqrt{7} & 0
\end{array}\right)
$$

$$
\rightsquigarrow \left(\begin{array}{ccccc|c}
-\sqrt{7} & 3i & 0 & 0 & i & 0 \\
-1 & -i\sqrt{7} & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
-i & -3i & 0 & 0 & -\sqrt{7} & 0
\end{array}\right)
\begin{array}{c}
\sqrt{7} \\ \\ \\ \\ i
\end{array}
$$

$\rightsquigarrow$ ...

$$\ldots \rightsquigarrow \left(\begin{array}{ccccc|c} 0 & -4i & 0 & 0 & i+\sqrt{7} & 0 \\ 1 & -i\sqrt{7} & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & \sqrt{7}-3i & 0 & 0 & -i-\sqrt{7} & 0 \end{array}\right) \quad +1$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} 0 & \sqrt{7}-7i & 0 & 0 & -2\sqrt{7} & 0 \\ 1 & -i\sqrt{7} & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & \sqrt{7}-3i & 0 & 0 & -i-\sqrt{7} & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} 0 & i\sqrt{7}-1 & 0 & 0 & 2 & 0 \\ 1 & -i\sqrt{7} & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & \sqrt{7}-3i & 0 & 0 & -i-\sqrt{7} & 0 \end{array}\right) \quad \begin{array}{l} +1 \\ \\ \\ \\ \cdot \frac{i+\sqrt{7}}{2} \end{array}$$

NR:

$$(i\sqrt{7}-1)\cdot\frac{i+\sqrt{7}}{2} = \frac{1}{2}\left(-\sqrt{7}-i+7i-\sqrt{7}\right) = 3i-\sqrt{7}$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} 0 & i\sqrt{7}-1 & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{array}\right) \rightsquigarrow \ldots$$

$$\cdots \rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{i\sqrt{7}-1} & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{2}{i\sqrt{7}-1} & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1+\frac{2}{i\sqrt{7}-1} & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{i\sqrt{7}-1} & 0 \\ 0 & 0 & 1 & 0 & \frac{2}{i\sqrt{7}-1} & 0 \\ 0 & 0 & 0 & 1 & \frac{2}{i\sqrt{7}-1} & 0 \end{array}\right)$$

$$\Rightarrow \quad E_A(-\sqrt{7}) = {}_{\mathbb{C}}\left\langle \begin{pmatrix} -i\sqrt{7}-1 \\ -2 \\ -2 \\ -2 \\ i\sqrt{7}-1 \end{pmatrix} \right\rangle$$

Orthonormalbasis von $E_A(\sqrt{7})$: $\left(\frac{1}{2\sqrt{7}} \begin{pmatrix} -i\sqrt{7}-1 \\ -2 \\ -2 \\ -2 \\ i\sqrt{7}-1 \end{pmatrix} \right)$

- Orthonormalbasis in $E_A(-\sqrt{7})$:

$$\left(\begin{array}{ccccc|c} \sqrt{7} & i & i & i & i & 0 \\ -i & \sqrt{7} & 0 & 0 & i & 0 \\ -i & 0 & \sqrt{7} & 0 & i & 0 \\ -i & 0 & 0 & \sqrt{7} & i & 0 \\ -i & -i & -i & -i & \sqrt{7} & 0 \end{array}\right) \begin{array}{l} \\ \\ -1 \\ -1 \\ \\ \end{array}$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} \sqrt{7} & i & i & i & i & 0 \\ -i & \sqrt{7} & 0 & 0 & i & 0 \\ 0 & -\sqrt{7} & \sqrt{7} & 0 & 0 & 0 \\ 0 & -\sqrt{7} & 0 & \sqrt{7} & 0 & 0 \\ -i & -i & -i & -i & \sqrt{7} & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} \sqrt{7} & 3i & 0 & 0 & i & 0 \\ 1 & i\sqrt{7} & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -i & -3i & 0 & 0 & \sqrt{7} & 0 \end{array}\right) \begin{array}{l} -\sqrt{7} \\ \\ \\ \\ i \end{array}$$

$\rightsquigarrow \ldots$

$$\ldots \rightsquigarrow \left(\begin{array}{ccccc|c} 0 & -4i & 0 & 0 & i+\sqrt{7} & 0 \\ 1 & i\sqrt{7} & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -\sqrt{7}-3i & 0 & 0 & -i+\sqrt{7} & 0 \end{array}\right) \quad +1$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} 0 & -\sqrt{7}-7i & 0 & 0 & 2\sqrt{7} & 0 \\ 1 & i\sqrt{7} & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -\sqrt{7}-3i & 0 & 0 & -i+\sqrt{7} & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} 0 & -i\sqrt{7}-1 & 0 & 0 & 2 & 0 \\ 1 & i\sqrt{7} & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -\sqrt{7}-3i & 0 & 0 & -i+\sqrt{7} & 0 \end{array}\right) \quad +1, \quad \cdot \frac{i-\sqrt{7}}{2}$$

NR

$$(-i\sqrt{7}-1)\cdot \frac{i-\sqrt{7}}{2}$$
$$= \frac{1}{2}\left(\sqrt{7} - i + 7i + \sqrt{7}\right)$$
$$= 3i + \sqrt{7}$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} 0 & -i\sqrt{7}-1 & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{array}\right) \rightsquigarrow \ldots$$

$$\ldots \rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{-i\sqrt{7}-1} & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{-i\sqrt{7}-1} & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{2}{-i\sqrt{7}-1} & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1+\frac{2}{-i\sqrt{7}-1} & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{-i\sqrt{7}-1} & 0 \\ 0 & 0 & 1 & 0 & \frac{2}{-i\sqrt{7}-1} & 0 \\ 0 & 0 & 0 & 1 & \frac{2}{-i\sqrt{7}-1} & 0 \end{array}\right)$$

$$\Rightarrow E_A(-\sqrt{7}) = {}_{\mathbb{C}}\left\langle \begin{pmatrix} i\sqrt{7}-1 \\ -2 \\ -2 \\ -2 \\ -i\sqrt{7}-1 \end{pmatrix} \right\rangle$$

Orthonormalbasis von $E_A(-\sqrt{7})$: $\left(\frac{1}{2\sqrt{7}} \begin{pmatrix} i\sqrt{7}-1 \\ -2 \\ -2 \\ -2 \\ -i\sqrt{7}-1 \end{pmatrix} \right)$

Mit der unitären Matrix

$$S = \frac{1}{2\sqrt{21}} \begin{pmatrix} 0 & 0 & 6 & -i\sqrt{21}-\sqrt{3} & i\sqrt{21}-\sqrt{3} \\ -\sqrt{42} & -\sqrt{14} & -2 & -2\sqrt{3} & -2\sqrt{3} \\ \sqrt{42} & -\sqrt{14} & -2 & -2\sqrt{3} & -2\sqrt{3} \\ 0 & 2\sqrt{14} & -2 & -2\sqrt{3} & -2\sqrt{3} \\ 0 & 0 & 6 & i\sqrt{21}-\sqrt{3} & -i\sqrt{21}-\sqrt{3} \end{pmatrix}$$

wird

$$\bar{S}^t A S = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \sqrt{7} & \\ & & & & -\sqrt{7} \end{pmatrix} =: D$$

Wir haben A unitär diagonalisiert.

Probe: $\bar{S}^t S = E_5$,

$$AS = SD$$