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<u>Bsp</u> zu Matrixexponentialfunktion

Sei $A = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix} \in \mathbb{C}^{3\times 3}$

Zu berechnen: $\exp(Ax)$ für $x \in \mathbb{R}$

Charakteristisches Polynom:

$$\chi_A(X) = \det(A - XE_3)$$

$$= \det \begin{pmatrix} 1-X & 2 & -2 \\ -2 & 1-X & 1 \\ -1 & 2 & -X \end{pmatrix}$$

$$= \det \begin{pmatrix} 1-X & 0 & -2 \\ -2 & 2-X & 1 \\ -1 & 2-X & -X \end{pmatrix} = \dots$$

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$$\ldots = (2-X)\det\begin{pmatrix} 1-X & 0 & -2 \\ -2 & 1 & 1 \\ -1 & 1 & -X \end{pmatrix}$$

$$= (2-X)\det\begin{pmatrix} 1-X & 0 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & -1-X \end{pmatrix}$$

$$= (2-X)\det\begin{pmatrix} 1-X & -2 \\ 1 & -1-X \end{pmatrix}$$

$$= (2-X)\left(-1-X+X+X^2+2\right)$$

$$= (2-X)\left(X^2+1\right)$$

$$= -(X-2)(X-i)(X-(-i))$$

Wir haben also ...

folgende Eigenwerte:

$\lambda_1 = 2 \qquad aV_A(2) = 1$

$\lambda_2 = i \qquad aV_A(i) = 1$

$\lambda_3 = -i \qquad aV_A(-i) = 1$

Insbesondere ist A diagonali-
sierbar.

zu $\lambda_1 = 2$:

$$A - 2E_3 = \begin{pmatrix} -1 & 2 & -2 \\ -2 & -1 & 1 \\ -1 & 2 & -2 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & -2 & 2 \\ 0 & -5 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$\Rightarrow$ Eigenraum $E_A(2)$

hat Basis $\left(\boxed{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \right)$

zu $\lambda_2 = i$:

$$A - iE_3 = \begin{pmatrix} 1-i & 2 & -2 \\ -2 & 1-i & 1 \\ -1 & 2 & -i \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & i \\ 0 & 4-2i & -i-3 \\ 0 & -3-i & 1+2i \end{pmatrix}$$

NR: $\frac{1+2i}{-3-i} = \frac{(1+2i)(-3+i)}{(-3-i)(-3+i)} = \frac{-5-5i}{10} = -\frac{1}{2} - \frac{i}{2}$

NR: $\frac{-i-3}{4-2i} = \frac{(-i-3)(4+2i)}{(4-2i)(4+2i)} = \frac{-10-10i}{20} = -\frac{1}{2} - \frac{i}{2}$

$$\sim \begin{pmatrix} 1 & -2 & i \\ 0 & 1 & -\frac{1}{2}-\frac{i}{2} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2}-\frac{i}{2} \end{pmatrix}$$

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$\Rightarrow$ Eigenraum $E_A(i)$

hat Basis $\left(\boxed{\begin{pmatrix} 1 \\ \frac{1}{2}+\frac{i}{2} \\ 1 \end{pmatrix}} \right)$.

Zu $\lambda_3 = -i$:

$$A + iE_3 = \begin{pmatrix} 1+i & 2 & -2 \\ -2 & 1+i & 1 \\ -1 & 2 & i \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & -i \\ 0 & 4+2i & i-3 \\ 0 & -3+i & 1-2i \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & -i \\ 0 & 1 & -\frac{1}{2}+\frac{i}{2} \end{pmatrix}$$

alles komplex konjugiert zum Fall $\lambda_2 = i$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2}+\frac{i}{2} \end{pmatrix}$$

$\Rightarrow$ Eigenraum $E_A(-i)$

hat Basis $\left(\boxed{\begin{pmatrix} 1 \\ \frac{1}{2}-\frac{i}{2} \\ 1 \end{pmatrix}}\right)$

$$\Rightarrow S = \begin{pmatrix} \boxed{\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ \frac{1}{2}+\frac{i}{2} \\ 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ \frac{1}{2}-\frac{i}{2} \\ 1 \end{matrix}} \end{pmatrix},$$

$$S^{-1}AS = \begin{pmatrix} 2 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} =: D$$

(auch "J" genannt)

Wir berechnen S^{-1}:

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & \frac{1}{2}+\frac{i}{2} & \frac{1}{2}-\frac{i}{2} & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \ldots$$

$$\ldots \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -\frac{1}{2}+\frac{i}{2} & -\frac{1}{2}-\frac{i}{2} & 0 & 1 & -1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -i & \frac{1}{2}-\frac{i}{2} & 1 & -1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2}-\frac{i}{2} & -i & i \\ 0 & 0 & 1 & \frac{1}{2}+\frac{i}{2} & i & -i \end{array}\right)$$

$$\underbrace{\begin{pmatrix} -1 & 0 & 1 \\ -\frac{1}{2}-\frac{i}{2} & -i & i \\ \frac{1}{2}+\frac{i}{2} & i & -i \end{pmatrix}}_{= S^{-1}}$$

Es ist

$$\exp(Dx) = \exp\left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \cdot x \right)$$

$$= \ldots$$

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$$\ldots = \begin{pmatrix} e^{2x} & 0 & 0 \\ 0 & e^{ix} & 0 \\ 0 & 0 & e^{-ix} \end{pmatrix}$$

$$\Rightarrow \exp(Ax)$$

$$= S \cdot \exp(Dx) \cdot S^{-1}$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & \frac{1}{2}+\frac{i}{2} & \frac{1}{2}-\frac{i}{2} \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{2x} & 0 & 0 \\ 0 & e^{ix} & 0 \\ 0 & 0 & e^{-ix} \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{2}-\frac{i}{2} & -i & i \\ \frac{1}{2}+\frac{i}{2} & i & -i \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{ix} & e^{-ix} \\ e^{2x} & \left(\frac{1}{2}+\frac{i}{2}\right)e^{ix} & \left(\frac{1}{2}-\frac{i}{2}\right)e^{-ix} \\ e^{2x} & e^{ix} & e^{-ix} \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{2}-\frac{i}{2} & -i & i \\ \frac{1}{2}+\frac{i}{2} & i & -i \end{pmatrix}$$

$$= \left(\begin{array}{l|l|l} \frac{1}{2}(e^{ix}+e^{-ix}) - \frac{i}{2}(e^{ix}-e^{-ix}) & -i\,(e^{ix}-e^{-ix}) & i(e^{ix}-e^{-ix}) \\ \hline -e^{2x} + \frac{1}{2}(e^{ix}+e^{-ix}) & -\frac{i}{2}(e^{ix}-e^{-ix}) + \frac{1}{2}(e^{ix}+e^{-ix}) & e^{2x} + \frac{i}{2}(e^{ix}-e^{-ix}) - \frac{1}{2}(e^{ix}+e^{-ix}) \\ \hline -e^{2x} + \frac{1}{2}(e^{ix}+e^{-ix}) - \frac{i}{2}(e^{ix}-e^{-ix}) & -i\,(e^{ix}-e^{-ix}) & e^{2x} + i(e^{ix}-e^{-ix}) \end{array}\right)$$

$$= \ldots$$

$$\ldots = \begin{pmatrix} \cos(x)+\sin(x) & 2\sin(x) & -2\sin(x) \\ -e^{2x}+\cos(x) & \sin(x)+\cos(x) & e^{2x}-\sin(x)-\cos(x) \\ -e^{2x}+\cos(x)+\sin(x) & 2\sin(x) & e^{2x}-2\sin(x) \end{pmatrix}$$

Probe: $\det\left(\exp(Ax)\right)$

etwas aufwendig in diesem Fall

$$= \det \begin{pmatrix} \cos(x)+\sin(x) & 2\sin(x) & -2\sin(x) \\ -e^{2x}+\cos(x) & \sin(x)+\cos(x) & e^{2x}-\sin(x)-\cos(x) \\ -e^{2x}+\cos(x)+\sin(x) & 2\sin(x) & e^{2x}-2\sin(x) \end{pmatrix}$$

$$= \det \begin{pmatrix} \cos(x)+\sin(x) & 2\sin(x) & -2\sin(x) \\ -e^{2x}+\cos(x) & \sin(x)+\cos(x) & e^{2x}-\sin(x)-\cos(x) \\ -e^{2x} & 0 & e^{2x} \end{pmatrix}$$

$$= e^{2x} \det \begin{pmatrix} \cos(x)+\sin(x) & 2\sin(x) & -2\sin(x) \\ -e^{2x}+\cos(x) & \sin(x)+\cos(x) & e^{2x}-\sin(x)-\cos(x) \\ -1 & 0 & 1 \end{pmatrix}$$

$$= e^{2x} \det \begin{pmatrix} \cos(x)-\sin(x) & 2\sin(x) & -2\sin(x) \\ -\sin(x) & \sin(x)+\cos(x) & e^{2x}-\sin(x)-\cos(x) \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \ldots$$

$$\dots = e^{2x} \det \begin{pmatrix} \cos(x) - \sin(x) & 2\sin(x) \\ -\sin(x) & \sin(x) + \cos(x) \end{pmatrix}$$

$$= e^{2x} \left(\cos(x)^2 - \sin(x)^2 + 2\sin(x)^2 \right)$$

$$= e^{2x}$$

$$\exp(\operatorname{tr}(Ax))$$

$$= \exp\left(\operatorname{tr}\left(\begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix} \cdot x \right) \right)$$

$$= \exp(2x) = e^{2x}$$

Das ist dasselbe, wie erhofft.