Bsp für Riccatische Differentialgleichung

Wir betrachten die Differentialgleichung

$$\boxed{y' = y^2 + 2y + 2}$$

(a) Man bestätige, daß
$y(x) = \tan(x) - 1$ (auf $\mathbb{R} \setminus \{\frac{\pi}{2} + \pi k : k \in \mathbb{Z}\}$)
eine Lösung ist.

(b) Man finde die Lösung zur Anfangsbedingung
$y(0) = 0$.

Zu (a):
$$y(x) = \tan(x) - 1$$
$$y'(x) = 1 + \tan(x)^2$$
$$y(x)^2 + 2y + 2 = \ldots$$

$$\ldots \left(\tan(x) - 1\right)^2 + 2\left(\tan(x) - 1\right) + 2$$

$$= \tan(x)^2 - \underline{2\tan(x)} + 1$$

$$+ \underline{2\tan(x)} - \underline{\underline{2}} + \underline{\underline{2}}$$

$$= \tan(x)^2 + 1$$

Das ist dasselbe.

Zu (b). Ansatz:

$$y = \tan(x) - 1 + \frac{1}{v}$$

$$y' = 1 + \tan(x)^2 - \frac{v'}{v^2}$$

$$y^2 + 2y + 2$$

$$= \left(\tan(x) - 1 + \frac{1}{v}\right)^2$$

$$+ 2\left(\tan(x) - 1 + \frac{1}{v}\right) + 2 = \ldots$$

$$\ldots \quad \tan(x)^2 + 1 + \frac{1}{v^2}$$

$$\underline{-2\tan(x)} + 2\tan(x)\cdot\frac{1}{v} \underline{\underline{- \frac{2}{v}}}$$

$$\underline{+2\tan(x)} \underline{\underline{-2}} \underline{\underline{+\frac{2}{v}}} \underline{\underline{+2}}$$

$$= \tan(x)^2 + 1 + \frac{1}{v^2} + 2\tan(x)\cdot\frac{1}{v}$$

Also wird $y' = y^2 + 2y + 2$ zu

$$1 + \tan(x)^2 - \frac{v'}{v^2} = \tan(x)^2 + 1 + \frac{1}{v^2} + 2\tan(x)\cdot\frac{1}{v} \; ,$$

also zu

$$v' = -2\tan(x)\, v - 1$$

Zunächst lösen wir

$$v' = -2\tan(x)\, v \, .$$

$$\left[\ln(|v|)\right] = \int \frac{1}{v}\, dv = \int \frac{v'}{v}\, dx$$

$$= \int -2\tan(x)\, dx$$

$$= 2 \int \frac{-\sin(x)}{\cos(x)}\, dx$$

$$= 2 \left[ \ln\left(|\cos(x)|\right) \right] dx$$

$$\Rightarrow \quad |v| = e^{2\ln\left(|\cos(x)|\right) + c}, \quad \text{wobei } c \in \mathbb{R}$$

$$\Rightarrow \quad v = \cos(x)^2 \cdot d, \quad \text{wobei } d \in \mathbb{R}.$$

Variation der Konstanten:

$$v = \cos(x)^2 \cdot d(x)$$

$$v' = -2\cos(x)\sin(x)\, d(x)$$

$$+ \cos(x)^2\, d'(x)$$

$$-2\tan(x)\, v = -2\tan(x)\cos(x)^2 d(x)$$

$$= -2\sin(x)\cos(x)\, d(x)$$

Zu lösen ist also:

$$-2\cos(x)\sin(x)\, d(x) + \cos(x)^2 d'(x)$$

$$= -2\sin(x)\cos(x)\, d(x) - 1$$

Also $d'(x) = -\dfrac{1}{\cos(x)^2} = \big(-\tan(x)\big)'$

Also $d(x) = -\tan(x) + s$, wobei $s \in \mathbb{R}$.

Also $v = v(x)$

$$= \cos(x)^2\big(-\tan(x) + s\big)$$

$$= -\cos(x)\sin(x) + s \cdot \cos(x)^2$$

$$= \cos(x)\big(-\sin(x) + s\cos(x)\big)$$

Also

$$y = y(x) = \tan(x) - 1 + \frac{1}{\cos(x)\,(-\sin(x) + s\cos(x))}$$

Anfangsbedingung verlangt:

$$0 = y(0) = \tan(0) - 1 + \frac{1}{\cos(0)\,(-\sin(0) + s\cos(0))}$$

$$= -1 + \frac{1}{s}$$

$\Rightarrow \quad s = 1$

Somit ist

$$\boxed{y = y(x) = \tan(x) - 1 + \frac{1}{\cos(x)\,(\cos(x) - \sin(x))}}$$

eine Lösung von $y' = y^2 + 2y + 2$ ...

... zur Anfangsbedingung

$y(0) = 0$.

Probe:

$$y' = 1 + \tan(x)^2 - \frac{-\sin(x)(\cos(x)-\sin(x)) + \cos(x)(-\sin(x)-\cos(x))}{\cos(x)^2(\cos(x)-\sin(x))^2}$$

$$= 1 + \tan(x)^2 - \frac{\sin(x)^2 - \overbrace{\cos(x)^2}^{=1-\sin(x)^2} - 2\sin(x)\cos(x)}{\cos(x)^2(\cos(x)-\sin(x))^2}$$

$$= 1 + \tan(x)^2 + \frac{-2\sin(x)^2 + 1 + 2\sin(x)\cos(x)}{\cos(x)^2(\cos(x)-\sin(x))^2}$$

$$y^2+2y+2 = \left(\tan(x) - 1 + \frac{1}{\cos(x)(\cos(x)-\sin(x))}\right)^2$$

$$+ 2\left(\tan(x) - 1 + \frac{1}{\cos(x)(\cos(x)-\sin(x))}\right)$$

$$+ 2$$

$$= \tan(x)^2 + 1 + \frac{1}{\cos(x)^2(\cos(x)-\sin(x))^2}$$

$$- 2\cancel{\tan}(x) + \frac{2\tan(x) - \cancel{2}}{\cos(x)(\cos(x)-\sin(x))}$$

$$+ \ldots$$

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$$\dots\quad 2\cancel{\tan}(x) - \cancel{2} + \frac{\cancel{2}}{\cancel{\cos(x)}\left(\cos(x)-\sin(x)\right)}$$

$$+ \cancel{2}$$

$$= \tan(x)^2 + 1$$

$$+ \frac{1 + 2\tan(x)\cos(x)\left(\cos(x)-\sin(x)\right)}{\cos(x)^2\left(\cos(x)-\sin(x)\right)^2}$$

$$= \tan(x)^2 + 1$$

$$+ \frac{1 + 2\sin(x)\cos(x) - 2\sin(x)^2}{\cos(x)^2\left(\cos(x)-\sin(x)\right)^2}$$

Paßt.