29.05.20-5

Bsp für Substitution

$$\int_{e}^{e^3} \frac{1}{x \ln(x)}\, dx = \int_{e}^{e^3} \frac{1}{\ln(x)} \cdot \frac{1}{x}\, dx$$

$$\boxed{\begin{array}{l} u(x) = \ln(x) \\ u'(x) = \frac{du}{dx} = \frac{1}{x} \end{array}}$$

$$\int_{x=e}^{x=e^3} \frac{1}{u} \frac{du}{dx}\, dx$$

$$= \int_{u=\ln(e)=1}^{u=\ln(e^3)=3} \frac{1}{u}\, du$$

$$= \Big[\ln(u) \Big]_{u=}^{u=3}$$

$$= \ln(3) - \ln(1)$$

$$= \ln(3)$$

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Bsp für trickreiche Substitution

$$\int \cos(\ln(x))\, dx$$

"Kamel-methode" $=$

$$\int \cos(\ln(x)) \cdot x \cdot \frac{1}{x}\, dx$$

den Faktor muß man haben, wenn man $u(x) = \ln(x)$ verwenden will

$u(x) = \ln(x)$

$u'(x) = \frac{du}{dx} = \frac{1}{x}$

$=$

$e^u = x$

$$\int \cos(u) \cdot e^u \cdot \frac{du}{dx}\, dx$$

$$= \int \underbrace{\cos(u)}_{f'(u)} \cdot \underbrace{e^u}_{g(u)}\, du$$

$$= \left[\underbrace{\sin(u)}_{f(u)}\, \underbrace{e^u}_{g(u)} \right] - \int \underbrace{\sin(u)}_{f(u)}\, \underbrace{e^u}_{g'(u)}\, du$$

$= \ldots$

$$\ldots = \left[\sin(u)\, e^{u}\right] - \left(\left[(-\cos(u))\, e^{u} \right] - \int -\cos(u)\, e^{u}\, du \right)$$

$$= \left[\sin(u)\, e^{u} + \cos(u)\, e^{u}\right] - \int \cos(u)\, e^{u}\, du$$

$$\Longrightarrow \quad 2 \int \cos(u)\, e^{u}\, du = \left[\sin(u)\, e^{u} + \cos(u)\, e^{u}\right]$$

$$\Longrightarrow \quad \int \cos(u)\, e^{u}\, du = \left[\frac{1}{2} e^{u} \left(\sin(u) + \cos(u) \right) \right]$$

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Zusammen:

$$\int \cos(\ln(x))\,dx$$

$$= \left[\frac{1}{2}e^{\ln(x)}\left(\sin(\ln(x)) + \cos(\ln(x))\right)\right]$$

$$= \left[\frac{1}{2}x\left(\sin(\ln(x)) + \cos(\ln(x))\right)\right]$$

Probe $\frac{d}{dx}\ \frac{1}{2}x\left(\sin(\ln(x)) + \cos(\ln(x))\right)$

$$= \frac{1}{2}\cdot 1\cdot\left(\sin(\ln(x)) + \cos(\ln(x))\right)$$

$$+\frac{1}{2}\cdot\cancel{x}\cdot\left(\cos(\ln(x))\cdot\frac{1}{\cancel{x}}\right.$$

$$\left.+(-\sin(\ln(x)))\cdot\frac{1}{\cancel{x}}\right)$$

$$= \cos(\ln(x))$$

Bsp für trickreiche Substitution

$$\int_0^1 \sqrt{1+u^2}\, du$$

Wir wollen $u(x) = \sinh(x)$

verwenden.

(Auf so eine Idee kommt man z.B.

durch einen Hinweis.)

Wir brauchen Integrationsgrenzen a, b

in der neuen Variablen x:

$$0 = u(a) = \sinh(a) \Rightarrow a = 0$$

$$1 = u(b) = \sinh(b) = \frac{1}{2}\left(e^b - e^{-b}\right)$$

$$\Rightarrow 2 = e^b - e^{-b}$$

...

$$\text{III} \Rightarrow \quad e^{b} - 2 - e^{-b} = 0$$

$$\Rightarrow \quad (e^{b})^2 - 2e^{b} - 1 = 0$$

|| quadratische Ergänzung

$$(e^{b} - 1)^2 - 2$$

$$\Rightarrow \quad \underbrace{e^{b} - 1}_{\geq -1} = \pm\sqrt{2}$$

Entscheidung für "+" erzwingen

$$\Rightarrow \quad e^{b} - 1 = +\sqrt{2}$$

$$\Rightarrow \quad e^{b} = 1 + \sqrt{2}$$

$$\Rightarrow \quad b = \ln(1 + \sqrt{2})$$

Also, mit rückwärts angewandter

Bemerkung:

$$\int_{u=0}^{u=1} \sqrt{1+u^2}\, du$$

$$= \int_{x=a=0}^{x=b=\ln(1+\sqrt{2})} \sqrt{1+u(x)^2}\, \frac{du}{dx}\, dx$$

$$= \int_0^{\ln(1+\sqrt{2})} \sqrt{1+\sinh(x)^2}\, \cosh(x)\, dx$$

hyperbolischer Pythagoras $\cosh(x)^2 - \sinh(x)^2 = 1$

$$= \int_0^{\ln(1+\sqrt{2})} \sqrt{\cosh(x)^2}\, \cosh(x)\, dx$$

$\cosh(x) \geq 0$

$$= \int_0^{\ln(1+\sqrt{2})} \cosh(x) \cdot \cosh(x)\, dx$$

$$= \int_0^{\ln(1+\sqrt{2})} \cosh(x)^2$$

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$$= \int_0^{\ln(1+\sqrt{2})} \left(\frac{1}{2}\left(e^{x} + e^{-x}\right)\right)^2 dx$$

$$= \int_0^{\ln(1+\sqrt{2})} \frac{1}{4}\left(e^{2x} + 2 + e^{-2x} \right) dx$$

$$= \left[\frac{1}{4}\left(\frac{1}{2} e^{2x} + 2x + \frac{1}{-2} e^{-2x} \right) \right]_0^{\ln(1+\sqrt{2})}$$

$$= \left(\frac{1}{8} e^{2\ln(1+\sqrt{2})} + \frac{1}{2}\ln(1+\sqrt{2}) - \frac{1}{8} e^{-2\ln(1+\sqrt{2})} \right)$$

$$= \frac{1}{8}(1+\sqrt{2})^2 + \frac{1}{2}\ln(1+\sqrt{2}) - \frac{1}{8}\frac{1}{(1+\sqrt{2})^2}$$

das würde so schon als Endergebnis gelten

$$= \ldots$$

$$\ldots = \frac{1}{8}\left(3+2\sqrt{2}\right) + \frac{1}{2}\ln\left(1+\sqrt{2}\right)$$

$$-\frac{1}{8}\,\frac{1\cdot\left(3-2\sqrt{2}\right)}{\left(3+2\sqrt{2}\right)\left(3-2\sqrt{2}\right)}$$

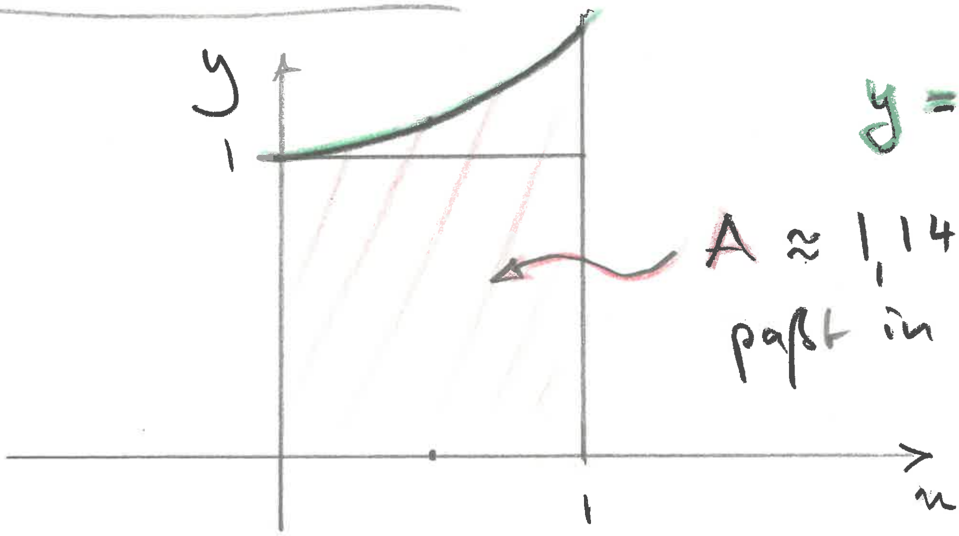
$$= \frac{1}{8}\left(3+2\sqrt{2}\right) + \frac{1}{2}\ln\left(1+\sqrt{2}\right)$$

$$-\frac{1}{8}\left(3-2\sqrt{2}\right)$$

$$= \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left(1+\sqrt{2}\right)$$

$$\approx 1{,}148$$

geometrische Probe:



$y = \sqrt{1+x^2}$

$A \approx 1{,}148$
paßt in etwa