Bsp

(1) $\int \underset{f}{\sin(x)} \underset{g'}{\cos(x)}\, dx$

$= \left[\underset{f}{\sin(x)}\ \underset{g}{\sin(x)} \right] - \int \underset{f'}{\cos(x)}\ \underset{g}{\sin(x)}\, dx$

$\Rightarrow \quad 2 \int \sin(x) \cos(x)\, dx$

$= \left[\sin(x)^2 \right]$

$\Rightarrow \quad \int \sin(x) \cos(x)\, dx = \left[\frac{1}{2} \sin(x)^2 \right]$

(2) $\int \sin(x) \cos(x)\, dx = ?$

NR: $\sin(x) \cos(x)$

$= \frac{1}{2i} \left(e^{ix} - e^{-ix} \right) \frac{1}{2} \left(e^{ix} + e^{-ix} \right)$

$= \ldots$

$$\ldots = \frac{1}{4i}\left(\left(e^{ix}\right)^2 - \left(e^{-ix}\right)^2 \right)$$

$$= \frac{1}{4i}\left(e^{2ix} - e^{-2ix} \right)$$

$$= \frac{1}{4i} \; 2i \sin(2x)$$

$$= \frac{1}{2} \sin(2x)$$

(Ergebnis NR: Additionstheorem rückwärts. Wenn das jemand von vornherein sieht, kann er abkürzen.)

Also:

$$\int \sin(x)\cos(x)\,dx$$

$$\overset{NR}{=} \int \frac{1}{2}\sin(2x)\,dx$$

NR 2: $\frac{d}{dx} \cos(2x) = -2\sin(2x)$

Also:

$$\int \sin(x)\cos(x)\,dx = \int \frac{1}{2}\sin(2x)\,dx$$

$$= -\frac{1}{4}\int -2\sin(2x)\,dx$$

$$\overset{NR2}{=} -\frac{1}{4}\left[\cos(2x)\right]$$

$$= \left[-\frac{1}{4}\cos(2x)\right]$$

(3) Vergleich der Ergebnisse:

Ist $\left[\frac{1}{2}\sin(x)^2\right] \overset{?}{=} \left[-\frac{1}{4}\cos(2x)\right]$

NR: $\frac{1}{2}\sin(x)^2 = \frac{1}{2}\left(\frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\right)^2$

$= \ldots$

$$\ldots = -\frac{1}{8}\left(e^{2ix} - 2 + e^{-2ix}\right)$$

$$= -\frac{1}{8}\left(2\cos(2x) - 2\right)$$

$$= -\frac{1}{4}\cos(2x) + \frac{1}{4}$$

Also:

$$\left[\frac{1}{2}\sin(x)^2\right] = \left[-\frac{1}{4}\cos(2x) + \frac{1}{4}\right]$$

$$= \left[-\frac{1}{4}\cos(2x)\right]$$

Konstante kann entfallen

Bsp

(1) $\int \underset{f'}{x}\,\underset{g}{\ln(x)}\,dx$

$$= \left[\underset{f}{\tfrac{1}{2}x^2}\cdot\underset{g}{\ln(x)}\right] - \int \underset{f}{\tfrac{1}{2}x^2}\cdot\underset{g'}{\tfrac{1}{x}}\,dx$$

$$= \left[\tfrac{1}{2}x^2\cdot\ln(x) - \tfrac{1}{4}x^2\right]$$

Probe:

$$\frac{d}{dx}\left(\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2\right)$$

$$= x\ln(x) + \frac{1}{2}x^2\cdot\frac{1}{x} - \frac{1}{2}x$$

$$= x\ln(x)$$

(2) $$\int \underbrace{x}_{f} \; \underbrace{\ln(x)}_{g'} \, dx$$

$$\overset{\text{Bsp in §4.7.4.1}}{=} \Big[\underbrace{x}_{f} \, \underbrace{(x \ln(x) - x)}_{g} \Big]$$

$$- \int \underbrace{1}_{f'} \cdot \underbrace{(x \ln(x) - x)}_{g} \, dx$$

$$\Rightarrow \; 2 \int x \ln(x) \, dx$$

$$= \Big[x^2 \ln(x) - x^2 + \frac{1}{2} x^2 \Big]$$

$$\Rightarrow \; \int x \ln(x) \, dx$$

$$= \Big[\frac{1}{2} x^2 \ln(x) - \frac{1}{4} x^2 \Big]$$

Das stimmt mit Resultat aus (1) überein.

Bsp Sei $x > 1$.

$$\int \frac{1}{x \ln(x)} \, dx$$

$$= \int \frac{1}{\ln(x)} \cdot \frac{1}{x} \, dx$$

$u = \ln(x)$

$\Rightarrow$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x} dx$

$$\int \frac{1}{u} \, du = \left[\ln(u) \right]$$

$$= \left[\ln(\ln(x)) \right]$$

Probe: $\frac{d}{dx} \ln(\ln(x))$

$$= \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

Bsp

$$\int \sin(x)\cos(x)\,dx$$

$u = \sin(x)$

$\frac{du}{dx} = \cos(x)$

$du = \cos(x)\,dx$

$$\int u\,du$$

$$= \left[\frac{1}{2}u^2\right]$$

$$= \left[\frac{1}{2}\sin(x)^2\right]$$

Das wissen wir schon von 04.06.21-1

Bsp $\int \sqrt{e^x+1}\,dx$

$u = \sqrt{e^x+1}$

$\frac{du}{dx} = \frac{1}{2\sqrt{e^x+1}} \cdot e^x$

$du = \frac{e^x}{2\sqrt{e^x+1}}\,dx$

$$\int u \cdot \frac{2\sqrt{e^x+1}}{e^x}\,\frac{e^x}{2\sqrt{e^x+1}}\,dx$$

$$= \ldots$$

$$\ldots = \int u \cdot \frac{2u}{u^2-1}\, du$$

$$= 2\int \frac{u^2}{u^2-1}\, du$$

$$= 2\int \frac{u^2-1+1}{u^2-1}\, du$$

$$= \int 2 + \frac{2}{u^2-1}\, du$$

NR: $\frac{1}{u-1} - \frac{1}{u+1}$

$$= \frac{(u+1) - (u-1)}{(u-1)(u+1)} = \frac{2}{u^2-1}$$

$$= \int 2 + \frac{1}{u-1} - \frac{1}{u+1}\, du$$

$$= \ldots$$

$$\ldots = \left[2u + \ln(|u-1|) - \ln(|u+1|) \right]$$

$$= \left[2\sqrt{e^x+1} + \ln\Big(\underbrace{\sqrt{e^x+1}-1}_{\geq 0}\Big) - \ln\Big(\underbrace{\sqrt{e^x+1}+1}_{\geq 0}\Big) \right]$$