Bsp $\displaystyle\int \frac{1}{x^3+x^2}\,dx = \,?$

$$x^3+x^2 = (x+1)^1 \cdot x^2$$

Ansatz:

$$\frac{1}{(x+1)^1 \cdot x^2} \overset{!}{=} \frac{a_{1,1}}{(x+1)^1} + \frac{a_{2,1}}{x^1} + \frac{a_{2,2}}{x^2}$$

mit Unbekannten $a_{1,1}$, $a_{2,1}$, $a_{2,2}$.

Durchmultiplizieren mit $(x+1)\,x^2$:

$$1 \overset{!}{=} a_{1,1} \cdot x^2 + a_{2,1} \cdot (x+1) \cdot x + a_{2,2} \cdot (x+1)$$

$= \ldots$

$$\ldots = a_{1,1} \cdot x^2 + a_{2,1} \cdot (x^2 + x) + a_{2,2} \cdot (x+1)$$

$$= (a_{1,1} + a_{2,1}) \cdot x^2$$

$$+ (a_{2,1} + a_{2,2}) \cdot x$$

$$+ \quad a_{2,2} \cdot 1$$

Koeffizientenvergleich:

bei x^2 : $0 \stackrel{!}{=} a_{1,1} + a_{2,1}$

bei x : $0 \stackrel{!}{=} a_{2,1} + a_{2,2}$

bei 1 : $1 \stackrel{!}{=} a_{2,2}$

$$\Rightarrow a_{2,2} = 1 \Rightarrow a_{2,1} = -1, \quad a_{1,1} = 1$$

Also

$$\frac{1}{x^3+x^2} = \frac{1}{(x+1)\cdot x^2}$$

$$= \frac{1}{x+1} - \frac{1}{x} + \frac{1}{x^2}$$

Also

$$\int \frac{1}{x^3+x^2}\, dx$$

$$= \int \frac{1}{x+1} - \frac{1}{x} + \frac{1}{x^2}\, dx$$

$$= \left[\ln(|x+1|) - \ln(|x|) - \frac{1}{x} \right]$$

optional

$$= \left[\ln\left(\left|\frac{x+1}{x}\right|\right) - \frac{1}{x} \right]$$

Bsp

$$\int \frac{4}{(x^2+1)^2}\,dx = ?$$

$$x^2+1 = (x-i)(x+i)$$

$$(x^2+1)^2 = (x-i)^2(x+i)^2$$

Ansatz:

$$\frac{4}{(x^2+1)^2} = \frac{4}{(x-i)^2(x+i)^2}$$

$$\stackrel{!}{=} \frac{a_{1,1}}{x-i} + \frac{a_{1,2}}{(x-i)^2}$$

$$+ \frac{a_{2,1}}{x+i} + \frac{a_{2,2}}{(x+i)^2}$$

Durchmultiplizieren mit

$$(x^2+1)^2 = (x-i)^2(x+i)^2 :$$

$$4 \overset{!}{=} a_{1,1}(x-i)(x+i)^2$$

$$+ a_{1,2}(x+i)^2$$

$$+ a_{2,1}(x-i)^2(x+i)$$

$$+ a_{2,2}(x-i)^2$$

$$= a_{1,1}(x^3+ix^2+x+i)$$

$$+ a_{1,2}(x^2+2ix-1)$$

$$+ a_{2,1}(x^3-ix^2+x-i)$$

$$+ a_{2,2}(x^2-2ix-1)$$

Koeffizientenvergleich:

$$x^3:\quad 0 = a_{1,1} + a_{2,1}$$
$$x^2:\quad 0 = i a_{1,1} + a_{1,2} - i a_{2,1} + a_{2,2}$$
$$x:\quad 0 = a_{1,1} + 2i a_{1,2} + a_{2,1} - 2i a_{2,2}$$
$$1:\quad 4 = i a_{1,1} - a_{1,2} - i a_{2,1} - a_{2,2}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ i & 1 & -i & 1 & 0 \\ 1 & 2i & 1 & -2i & 0 \\ i & -1 & -i & -1 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ i & 1 & -i & 1 & 0 \\ 0 & 2i & 0 & -2i & 0 \\ 0 & -2 & 0 & -2 & 4 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2i & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -2i & 2 & 0 \\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -i \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & i \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow \begin{array}{l} a_{1,1} = -i,\quad a_{1,2} = -1, \\ a_{2,1} = i,\quad a_{2,2} = -1 \end{array}$$

$$\Rightarrow \frac{4}{(x^2+1)^2} = \frac{-i}{x-i} + \frac{-1}{(x-i)^2}$$

$$+ \frac{i}{x+i} + \frac{-1}{(x+i)^2}$$

$$\overset{\text{Trick}}{=} \frac{-i(x+i)+i(x-i)}{(x-i)(x+i)}$$

$$- \frac{1}{(x-i)^2} - \frac{1}{(x+i)^2}$$

$$= \frac{2}{x^2+1} - \frac{1}{(x-i)^2} - \frac{1}{(x+i)^2}$$

$$\Rightarrow \int \frac{4}{(x^2+1)^2}\,dx$$

$$= \int \frac{2}{x^2+1} - \frac{1}{(x-i)^2} - \frac{1}{(x+i)^2}\,dx = \dots$$

$$\ldots = \left[2\arctan(x) + \frac{1}{x-i} + \frac{1}{x+i}\right]$$

$$= \left[2\arctan(x) + \frac{(x+i)+(x-i)}{x^2+1}\right]$$

$$= \left[2\arctan(x) + \frac{2x}{x^2+1}\right]$$

(Den Trick müssen wir noch zur Methode ausbauen.)

Probe: $\frac{d}{dx}\left(2\arctan(x) + \frac{2x}{x^2+1}\right)$

$$= \frac{2}{x^2+1} + \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2}$$

$$= \frac{2(x^2+1) + 2(x^2+1) - 4x^2}{(x^2+1)^2} = \frac{4}{(x^2+1)^2}$$