Bsp

Sei $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

Wir wollen $\exp(Ax)$ bestimmen.

- $\chi_A(X) = \det \begin{pmatrix} -X & 1 & 0 \\ 0 & -X & 1 \\ 1 & 0 & -X \end{pmatrix}$

$$= \det \begin{pmatrix} -X & 1 & 0 \\ -X^2 & 0 & 1 \\ 1 & 0 & -X \end{pmatrix}$$

$$= -\det \begin{pmatrix} -X^2 & 1 \\ 1 & -X \end{pmatrix} = -(X^3 - 1)$$

$$= -(X-1)(X^2+X+1)$$

$$= -(X-1)\left(X+\frac{1}{2}-\frac{i}{2}\sqrt{3}\right)\left(X+\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)$$

$$\Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = -\frac{1}{2} + \frac{i}{2}\sqrt{3} =: \zeta$$

$$\lambda_3 = -\frac{1}{2} - \frac{i}{2}\sqrt{3} = \overline{\zeta}$$

$\Rightarrow$ A diagonalisierbar

$$\underbrace{\qquad\qquad}_{\substack{\overline{\zeta} = -\zeta - 1 \\ \zeta^2 + \zeta + 1 = 0}}$$

- $\lambda_1 = 1$: $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \text{ ist Basis von } E_A(1)$$

- $\lambda_2 = \zeta$: $\begin{pmatrix} -\zeta & 1 & 0 \\ 0 & -\zeta & 1 \\ 1 & 0 & -\zeta \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 0 & -\zeta \\ 0 & 1 & -\overline{\zeta} \end{pmatrix}$$

$$\Rightarrow \left(\begin{pmatrix} \zeta \\ \overline{\zeta} \\ 1 \end{pmatrix} \right) \text{ ist Basis von } E_A(\zeta)$$

- $\lambda_3 = \bar{\zeta}$: $\begin{pmatrix} -\bar{\zeta} & 1 & 0 \\ 0 & -\bar{\zeta} & 1 \\ 1 & 0 & -\bar{\zeta} \end{pmatrix}$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & -\bar{\zeta} \\ 0 & 1 & -\zeta \end{pmatrix}$$

$\Rightarrow \left(\begin{pmatrix} \bar{\zeta} \\ \zeta \\ 1 \end{pmatrix} \right)$ ist Basis von $E_A(\bar{\zeta})$

-

- Mit $S := \begin{pmatrix} 1 & \zeta & \bar{\zeta} \\ 1 & \bar{\zeta} & \zeta \\ 1 & 1 & 1 \end{pmatrix}$

wird $S^{-1}AS = \begin{pmatrix} 1 & & \\ & \zeta & \\ & & \bar{\zeta} \end{pmatrix} =: D$

Wir invertieren S :

$$\left(\begin{array}{ccc|ccc} 1 & \zeta & \bar{\zeta} & 1 & 0 & 0 \\ 1 & \bar{\zeta} & \zeta & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \rightsquigarrow \ldots$$

$$\ldots \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & -\frac{1}{2}+\frac{i}{2}\sqrt{3} & -\frac{1}{2}-\frac{i}{2}\sqrt{3} & 1 & 0 & 0 \\ 1 & -\frac{1}{2}-\frac{i}{2}\sqrt{3} & -\frac{1}{2}+\frac{i}{2}\sqrt{3} & 0 & 1 & 0 \end{array}\right) \begin{array}{l} \\ \\ \xleftarrow{+1} \; \xleftarrow{+1} \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & -\frac{1}{2}+\frac{i}{2}\sqrt{3} & -\frac{1}{2}-\frac{i}{2}\sqrt{3} & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & -\frac{1}{2}+\frac{i}{2}\sqrt{3} & -\frac{1}{2}-\frac{i}{2}\sqrt{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{i}{2}\sqrt{3} & -\frac{i}{2}\sqrt{3} & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & -i\sqrt{3} & \frac{1}{2}+\frac{i}{2}\frac{\sqrt{3}}{3} & -\frac{1}{2}+\frac{i}{2}\frac{\sqrt{3}}{3} & -i\frac{\sqrt{3}}{3} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{i}{2\sqrt{3}}-\frac{1}{6} & \frac{i}{2\sqrt{3}}-\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{i}{2\sqrt{3}}-\frac{1}{6} & -\frac{i}{2\sqrt{3}}-\frac{1}{6} & \frac{1}{3} \end{array}\right)$$

$$\Rightarrow \quad S^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 & 2 \\ -i\sqrt{3}-1 & i\sqrt{3}-1 & 2 \\ i\sqrt{3}-1 & -i\sqrt{3}-1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ \bar{\zeta} & \zeta & 1 \\ \zeta & \bar{\zeta} & 1 \end{pmatrix}$$

Also $\exp(Dx) = \exp\left(\begin{pmatrix} 1 & & \\ & \zeta & \\ & & \bar{\zeta} \end{pmatrix} x\right)$

$$= \begin{pmatrix} e^{x} & 0 & 0 \\ 0 & e^{\zeta x} & 0 \\ 0 & 0 & e^{\bar{\zeta} x} \end{pmatrix}$$

Also $\exp(Ax) = \exp(SDxS^{-1})$

$$= S \exp(Dx) S^{-1}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & \zeta & \bar{\zeta} \\ 1 & \bar{\zeta} & \zeta \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} e^{x} & 0 & 0 \\ 0 & e^{\zeta x} & 0 \\ 0 & 0 & e^{\bar{\zeta} x} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ \bar{\zeta} & \zeta & 1 \\ \zeta & \bar{\zeta} & 1 \end{pmatrix}$$

$$= \ldots$$

$$\ldots = \frac{1}{3}\begin{pmatrix} 1 & \zeta & \bar\zeta \\ 1 & \bar\zeta & \zeta \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} e^x & e^x & e^x \\ e^{\zeta x}\bar\zeta & e^{\zeta x}\zeta & e^{\zeta x} \\ e^{\bar\zeta x}\zeta & e^{\bar\zeta x}\bar\zeta & e^{\bar\zeta x} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} e^x + e^{\zeta x} + e^{\bar\zeta x} & e^x + \zeta^2 e^{\zeta x} + \bar\zeta^2 e^{\zeta x} & e^x + \zeta e^{\zeta x} + \bar\zeta e^{\bar\zeta x} \\ e^x + \bar\zeta^2 e^{\zeta x} + \zeta^2 e^{\bar\zeta x} & e^x + e^{\zeta x} + e^{\bar\zeta x} & e^x + \bar\zeta e^{\zeta x} + \zeta e^{\bar\zeta x} \\ e^x + \bar\zeta e^{\zeta x} + \zeta e^{\bar\zeta x} & e^x + \zeta e^{\zeta x} + \bar\zeta e^{\bar\zeta x} & e^x + e^{\zeta x} + e^{\bar\zeta x} \end{pmatrix}$$

N2 : (1)

$$e^{\zeta x} + e^{\bar\zeta x}$$

$$= e^{-\frac{x}{2} + \frac{i}{2}\sqrt{3}x} + e^{-\frac{x}{2} - \frac{i}{2}\sqrt{3}x}$$

$$= e^{-\frac{x}{2}} \cdot 2\cos\left(\frac{x\sqrt{3}}{2}\right)$$

(2)

$$\zeta e^{\zeta x} + \bar\zeta e^{\bar\zeta x}$$

$$= e^{\zeta x + \frac{2\pi i}{3}} + e^{\bar\zeta x - \frac{2\pi i}{3}}$$

$$= e^{-\frac{x}{2} + \frac{i}{2}\sqrt{3}x + \frac{2\pi i}{3}} + e^{-\frac{x}{2} - \frac{i}{2}\sqrt{3}x - \frac{2\pi i}{3}}$$

$$= e^{-\frac{x}{2}} \cdot 2\cos\left(\frac{x\sqrt{3}}{2} + \frac{2\pi}{3}\right)$$

(3)

$$\bar\zeta e^{\zeta x} + \zeta e^{\bar\zeta x}$$

$$= e^{\zeta x - \frac{2\pi i}{3}} + e^{\bar\zeta x + \frac{2\pi i}{3}}$$

$$= e^{-\frac{x}{2} + \frac{i}{2}\sqrt{3} - \frac{2\pi i}{3}} + e^{-\frac{x}{2} - \frac{i}{2}\sqrt{3} - \frac{2\pi i}{3}}$$

$$= e^{-\frac{x}{2}} \cdot 2\cos\left(\frac{x\sqrt{3}}{2} - \frac{2\pi}{3}\right)$$

Also

$$\exp(Ax) = \frac{2e^{-\frac{x}{2}}}{3}\begin{pmatrix} \cos\left(\frac{x\sqrt{3}}{2}\right) & \cos\left(\frac{x\sqrt{3}}{2}-\frac{2\pi}{3}\right) & \cos\left(\frac{x\sqrt{3}}{2}+\frac{2\pi}{3}\right) \\ \cos\left(\frac{x\sqrt{3}}{2}+\frac{2\pi}{3}\right) & \cos\left(\frac{x\sqrt{3}}{2}\right) & \cos\left(\frac{x\sqrt{3}}{2}-\frac{2\pi}{3}\right) \\ \cos\left(\frac{x\sqrt{3}}{2}-\frac{2\pi}{3}\right) & \cos\left(\frac{x\sqrt{3}}{2}+\frac{2\pi}{3}\right) & \cos\left(\frac{x\sqrt{3}}{2}\right) \end{pmatrix}$$

$$+ \frac{e^x}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{2e^{-\frac{x}{2}}}{3}\cos\left(\frac{x\sqrt{3}}{2}\right)\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$+ \frac{2e^{-\frac{x}{2}}}{3}\cos\left(\frac{x\sqrt{3}}{2}+\frac{2\pi}{3}\right)\cdot\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$+ \frac{2e^{-\frac{x}{2}}}{3}\cos\left(\frac{x\sqrt{3}}{2}-\frac{2\pi}{3}\right)\cdot\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$+ \frac{e^x}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

<u>Bsp</u> Wir wollen die Lösungen des Differentialgleichungssystems

$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = y_1$$

auf $\mathbb{R}$ finden zum

Anfangswert $y(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Wir schreiben das System:

$$\underbrace{\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix}}_{y'} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{y}$$

Die Lösungen sind von der Form

$$y(x) = \exp(Ax) \cdot c$$

mit $c \in \mathbb{R}^{3\times 1}$.

Anfangswertbedingung:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \stackrel{!}{=} y(0) = \underbrace{\exp(A \cdot 0)}_{= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \cdot c = c$$

Also haben wir die Lösung

$$y(x) = \exp(Ax) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\stackrel{\text{s. Bsp. oben}}{=} \frac{2e^{-\frac{x}{2}}}{3} \begin{pmatrix} \cos\left(\frac{x\sqrt{3}}{2}\right) \\ \cos\left(\frac{x\sqrt{3}}{2} + \frac{2\pi}{3}\right) \\ \cos\left(\frac{x\sqrt{3}}{2} - \frac{2\pi}{3}\right) \end{pmatrix} + \frac{e^x}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Probe!

$$y'(x) = -\frac{e^{-\frac{x}{2}}}{3}\begin{pmatrix}\cos\left(\frac{x\sqrt{3}}{2}\right)\\ \cos\left(\frac{x\sqrt{3}}{2}+\frac{2\pi}{3}\right)\\ \cos\left(\frac{x\sqrt{3}}{2}-\frac{2\pi}{3}\right)\end{pmatrix} + \frac{2e^{-\frac{x}{2}}}{3}\begin{pmatrix}-\sin\left(\frac{x\sqrt{3}}{2}\right)\frac{\sqrt{3}}{2}\\ -\sin\left(\frac{x\sqrt{3}}{2}+\frac{2\pi}{3}\right)\frac{\sqrt{3}}{2}\\ -\sin\left(\frac{x\sqrt{3}}{2}-\frac{2\pi}{3}\right)\frac{\sqrt{3}}{2}\end{pmatrix}$$

$$+\frac{e^{x}}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$A\,y(x) = \frac{2e^{-\frac{x}{2}}}{3}\begin{pmatrix}\cos\left(\frac{x\sqrt{3}}{2}+\frac{2\pi}{3}\right)\\ \cos\left(\frac{x\sqrt{3}}{2}-\frac{2\pi}{3}\right)\\ \cos\left(\frac{x\sqrt{3}}{2}\right)\end{pmatrix} + \frac{e^{x}}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

NR: (1) $LS = -\frac{1}{3}\cos\left(\frac{x\sqrt{3}}{2}\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{x\sqrt{3}}{2}\right)$

$$RS = \frac{2}{3}\cos\left(\frac{x\sqrt{3}}{2}+\frac{2\pi}{3}\right)$$

$$= \frac{2}{3}\left(\cos\left(\frac{x\sqrt{3}}{2}\right)\underbrace{\cos\left(\frac{2\pi}{3}\right)}_{-\frac{1}{2}} - \sin\left(\frac{x\sqrt{3}}{2}\right)\underbrace{\sin\left(\frac{2\pi}{3}\right)}_{\frac{1}{2}\sqrt{3}}\right)$$

(2), (3): Genauso, nur $\frac{2\pi}{3}$ resp. $-\frac{2\pi}{3}$ verschoben.

Und $y(0) = \frac{2}{3}\begin{pmatrix}1\\ \cos\left(\frac{2\pi}{3}\right)\\ \cos\left(-\frac{2\pi}{3}\right)\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{2}{3}\begin{pmatrix}1\\ -\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix}$$

Paßt